

Performance of the finite volume method in solving regularised Bingham flows: inertia effects in the lid-driven cavity flow

Alexandros Syrakos^{a,b,*}, Georgios C. Georgiou^{a,b}, Andreas N. Alexandrou^c

^a*Department of Mathematics and Statistics, University of Cyprus, PO Box 20537, 1678 Nicosia, Cyprus*

^b*Oceanography Centre, University of Cyprus, PO Box 20537, 1678 Nicosia, Cyprus*

^c*Department of Mechanical and Manufacturing Engineering, University of Cyprus, PO Box 20537, 1678 Nicosia, Cyprus*

Abstract

We extend our recent work on the creeping flow of a Bingham fluid in a lid-driven cavity, to the study of inertial effects, using a finite volume method and the Papanastasiou regularisation of the Bingham constitutive model [J. Rheology 31 (1987) 385-404]. The finite volume method used belongs to a very popular class of methods for solving Newtonian flow problems, which use the SIMPLE algorithm to solve the discretised set of equations, and have matured over the years. By regularising the Bingham constitutive equation it is easy to extend such a solver to Bingham flows since all that this requires is to modify the viscosity function. This is a tempting approach, since it requires minimum programming effort and makes available all the existing features of the mature finite volume solver. On the other hand, regularisation introduces a parameter which controls the error in addition to the grid spacing, and makes it difficult to locate the yield surfaces. Furthermore, the equations become stiffer and more difficult to solve, while the discontinuity at the yield surfaces causes large truncation errors. The present work attempts to investigate the strengths and weaknesses of such a method by applying it to the lid-driven cavity problem for a range of Bingham and Reynolds numbers (up to 100 and 5000 respectively). By employing techniques such as multigrid, local grid refinement, and an extrapolation procedure to reduce the effect of the regularisation parameter on the calculation of the yield surfaces (Liu et al. J. Non-Newtonian Fluid Mech. 102 (2002) 179-191), satisfactory results are obtained, although the weaknesses of the method become more noticeable as the Bingham number is increased.

Keywords: Finite volume method, Bingham flow, regularisation, lid-driven cavity, adaptive mesh refinement, truncation error, multigrid, SIMPLE

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1. Introduction

Finite volume methods (see [1] for a detailed description) are popular for the solution of fluid flow problems. They divide the computational domain into a number of small volumes, and integrate the governing differential equations over each volume. The resulting integrals, which consist of the fluxes

*Corresponding author

Email addresses: syrakos.alexandros@ucy.ac.cy (Alexandros Syrakos), georgios@ucy.ac.cy (Georgios C. Georgiou), andalexa@ucy.ac.cy (Andreas N. Alexandrou)

and source terms of the relevant physical quantities on each volume, are approximated by algebraic expressions involving the values of the dependent variables at selected points of the domain, such as the volume centres. This results in a system of non-linear algebraic equations, the solution of which gives approximate values of the dependent variables at the selected points. In the literature one can find that many different algebraic solvers have been used to solve this system, but the most popular appear to be SIMPLE [2] and its variants. These are iterative methods which consecutively solve a series of linear systems in each iteration: one linear system for each momentum equation, and a linear system for pressure which tries to enforce satisfaction of the continuity equation by relating velocity corrections to pressure corrections. Although SIMPLE is an old algorithm, it is still very popular and it has been used successfully in numerous studies to solve complex flow phenomena, [3] being a recent example involving non-Newtonian flow. SIMPLE is a slowly converging algorithm, but its performance can be greatly enhanced by using it in a multigrid context [4, 5]. The popularity of SIMPLE and its variants is due in part to the fact that they allow easy extension of the solver to account for additional physical phenomena: for each additional differential equation to be solved, the solution of a corresponding linear system is added to each SIMPLE iteration. Over the years, codes which implement this method have been developed and expanded to include many features, such as meshing capabilities, discretisation schemes, boundary condition choices, graphical user interfaces, turbulence models, and choices of models for different physical phenomena (heat transfer, combustion, chemical reactions, phase change, flow with a free surface etc.). In the present work we investigate the ability of such a solver to solve viscoplastic flows. This would be desirable, as it would greatly reduce the programming effort and it would allow one to also use other features of the solver, contrary to developing a specialised solver from scratch.

Viscoplastic materials behave as solids at low stress levels, but flow when the stress exceeds a critical value, the *yield stress*, τ_y . Suspensions of particles or macromolecules, such as pastes, gels, foams, drilling fluids, food products, and nanocomposites, are typical viscoplastic fluids. The simplest model for a viscoplastic material is the Bingham model, which exhibits a linear stress to rate-of-strain relationship during flow. The constant of proportionality of this linear relationship is called the *plastic viscosity*, μ . Thus, the Bingham constitutive equation is written as follows:

$$\begin{cases} \dot{\gamma} = \mathbf{0}, & \tau \leq \tau_y \\ \tau = \left(\frac{\tau_y}{\dot{\gamma}} + \mu \right) \dot{\gamma}, & \tau > \tau_y \end{cases} \quad (1)$$

where τ is the stress tensor and $\dot{\gamma}$ is the rate-of-strain tensor, $\dot{\gamma} \equiv \nabla \mathbf{u} + (\nabla \mathbf{u})^T$, \mathbf{u} being the velocity vector. The magnitudes of these two tensors are $\tau \equiv (\frac{1}{2} \tau : \tau)^{1/2}$ and $\dot{\gamma} \equiv (\frac{1}{2} \dot{\gamma} : \dot{\gamma})^{1/2}$.

Thus the Bingham constitutive equation has two branches, with different physical laws applying to yielded and unyielded areas. The solution of Bingham flows is difficult because the location of the interface between the yielded and unyielded regions is a priori unknown. The most popular approaches to tackle this problem fall into two categories. One category includes methods which approximate equation (1) by a regularised constitutive equation, which treats the whole material as a fluid of variable viscosity and is applicable throughout. The unyielded regions are approximated by locally assigning a very high value to the viscosity. This category includes methods such as that proposed by Bercovier and Engelman [6], and that by Papanastasiou [7]. A related method (although not strictly a regularisation method since it does not remove the discontinuity) is the bi-viscosity method of O'Donovan and Tanner [8], which treats the unyielded material as a separate, very viscous fluid. Regularisation methods make use of a parameter which, depending on the model, should be given a very large or a very small value, in order for the results to be a good approximation to the actual Bingham flow solution. It can be shown (e.g. [9, 10]) that in the limit when this parameter approaches asymptotically to infinity or zero, the

velocity field obtained with the regularisation method converges to that of the actual Bingham flow. However, in practice, the range of usable values for the regularisation parameter is limited by the fact that using extreme values causes numerical problems. A disadvantage of the regularisation approach is that since the whole material is a fluid, yield surfaces are not clearly defined. Usually the yield surfaces are approximated by the surfaces where the magnitude of the stress equals the yield stress. On the other hand, to implement such a method in an existing solver one only has to define appropriately the function which calculates the viscosity. This makes regularisation methods very suitable for the use examined in this work, that is for enabling existing solvers to tackle viscoplastic flows. In fact many commercial solvers have built-in or allow user-defined viscosity functions, and there exist a number of published works where commercial finite volume (e.g. in [11–14]) or finite element packages (e.g. in [15–17, 14]) were used in this way, applying either a regularisation method or the bi-viscosity method to solve viscoplastic flows.

The other category includes methods which start by deriving a variational inequality whose solution is equivalent to the solution of the original problem (variational inequality formulation of the problem). Their solution relies on the use of multiplier functions (projection methods and augmented Lagrangian methods) [18, 19, 10, 20]. These methods avoid the use of regularisation and thus solve the actual problem directly. They are mathematically more involved and the concepts may be difficult to follow for people who do not have the relevant mathematical background. Unfortunately, contrary to regularisation methods, it is not straightforward to incorporate them into existing general purpose solvers, and thus are not suitable for the present investigation. They are implemented usually in combination with a finite element discretisation scheme, but a finite volume scheme can also be used [21]. The availability of these techniques has led some researchers to support that regularisation methods should be avoided. However, apart from the aforementioned practical advantages of regularisation methods, one should also keep in mind that the Bingham and other viscoplastic models only approximate the behaviour of real materials. There is some ambiguity concerning the manifestation of yield stress by actual materials (see [22] or [23] for discussions and further references) and some researchers have suggested that regularised constitutive equations may in fact be better representations of the physical reality [24]. In any case, there is a large literature with successful application of regularisation methods to a wide range of problems - see [20] for some references. The results produced by regularisation methods should be interpreted with some caution though as pointed out in [9], because it is not easy to know when the regularisation parameter is large enough. A direct comparison between Papanastasiou regularisation and the augmented Lagrangian method is made in a recent publication [25].

The present work aims to explore the potential and the limitations of a finite volume method that uses the SIMPLE algebraic solver, together with a regularised constitutive equation, for solving viscoplastic flows. This was partly explored also in our previous publication [26], but the present work differs in the following: a) non-zero Reynolds number flows are also treated; b) the detrimental effect of the discontinuity exhibited at the yield surfaces on the accuracy of the finite volume solution is studied in more depth, by calculating the truncation error generated, and local grid refinement is proposed as a treatment; and c) more attention is given to the accurate determination of the yield surfaces, and ways to increase this accuracy are investigated.

Adaptive mesh refinement for viscoplastic flows was applied successfully in a Finite Element context already by Roquet and Saramito [27] and Zhang [28], with an augmented Lagrangian method in both cases, with the aim of improving the accuracy of the calculated yield surfaces. The present work employs a Finite Volume variant of adaptive mesh refinement, proposed in [29, 30]. The focus is not restricted to the yield surfaces, but the advantages of local grid refinement are assessed with respect to the overall flow field accuracy as well.

We employ the Papanastasiou [7] regularisation of equation (1). It introduces an exponential term

to replace the discontinuous constitutive equation (1) by a single equation, applicable throughout the material:

$$\boldsymbol{\tau} = \left[\frac{\tau_y}{\dot{\gamma}} \{1 - \exp(-m\dot{\gamma})\} + \mu \right] \dot{\boldsymbol{\gamma}} \quad (2)$$

where m is a stress growth parameter, which is required to be “sufficiently” large so that the ideal Bingham behaviour is approximated with satisfactory accuracy.

The method is tested on the square lid-driven cavity problem, which is probably the most popular problem for testing numerical methods in computational fluid dynamics. Consider a square cavity of side L , filled with a fluid which is set to motion by the lid of the cavity which moves with a tangential velocity U . In the case of Bingham flow, the flow field is characterised by two dimensionless numbers: the Reynolds number, which is defined in terms of the plastic viscosity μ ,

$$Re \equiv \frac{\rho U L}{\mu} \quad (3)$$

and the Bingham number Bn , defined as

$$Bn \equiv \frac{\tau_y L}{\mu U} \quad (4)$$

The two-dimensional Newtonian lid-driven cavity problem has been studied numerically in great detail - see e.g. the review papers [31, 32]. Accurate numerical results can be found in the literature for Reynolds numbers up to 10,000 [33, 34, 30]. The few available experimental studies [35–37] have shown that, even at moderate Reynolds numbers, the flow field is in fact three-dimensional. Three-dimensional computational studies have confirmed this and provided a value of $Re \approx 785$ as a critical value beyond which 3-dimensional features appear [38, 39]. Nevertheless, the two-dimensional lid-driven cavity problem, although unrealistic beyond a critical Reynolds number, is very attractive for testing numerical methods, as it is easy to set up due to the simple geometry and boundary conditions, and the flow field is complex enough to provide a good test for the numerical method.

Flow of viscoplastic fluids in a lid-driven cavity has also been studied numerically, although in the majority of cases the problem was only used as a validation test case and limited results were obtained. Most studies only considered creeping flow or flow at very low Reynolds numbers, such as the works of Bercovier and Engelman [6] and of Mitsoulis and Zisis [40], who used regularisation methods, and the works of Sanchez [41], Yu and Wachs [42], Muravleva and Olshanskii [43], Zhang [28] and Glowinski and Wachs [20], who used the augmented Lagrangian method. The results of all these studies agree qualitatively, in that two unyielded zones appear in general: one at the bottom of the cavity which is motionless, and one near the vortex centre which moves with solid body rotation. However, it is rather surprising that there is not an agreement on the precise shape of these zones, even among the augmented Lagrangian studies. A possible explanation is that, as noted by Glowinski and Wachs [20], the solution of the augmented Lagrangian method only becomes equivalent to the original Bingham problem when iterations have fully converged, but in any numerical procedure convergence is assumed when a certain non-zero tolerance has been reached. So, the accuracy depends on the choice of this tolerance, in a similar fashion that the accuracy of regularisation methods depends on the choice of regularisation parameter. To this one has to add the errors generated by the discretisation procedure. Our previous results [26] are closer to those of Mitsoulis and Zisis [40], and Glowinski and Wachs [20].

There are also some studies containing results for moderate or high Reynolds numbers. We note the works of Neophytou [44], Elias et al. [45], Frey et al. [46], Prashant and Derksen [47] and dos Santos et al. [48] who used regularisation methods or the bi-viscosity method, and the work of Vola et al. [49]

who used an augmented Lagrangian method. It is noteworthy that whereas in the creeping flow case the multipliers methods are dominant, in the inertia case the situation is reversed.

Concerning the discretisation methods, it appears that the finite element method is the usual choice. The finite volume method has been used in only two of the aforementioned studies, those of Neophytou [44] and Glowinski and Wachs [20]. The method of Neophytou [44] is a finite volume method which employs the SIMPLE algebraic solver and uses the Papanastasiou regularisation, and thus is of the category of methods examined here. It does not use more advanced techniques such as multigrid and local grid refinement, and unfortunately the results are limited to very low Bingham numbers, $Bn \leq 1$. The finite volume method of Glowinski and Wachs [20] on the other hand is adapted to the needs of the augmented Lagrangian method, and deals only with creeping flow.

The aim of the present work is to investigate the capabilities and limitations of the popular finite volume / SIMPLE method coupled with the Papanastasiou regularisation, by applying it to the simulation of Bingham flow in a lid driven cavity. Where limitations are detected, an effort is made to overcome them using techniques that are relatively easy to apply in the framework of such a method. We provide results for a wide range of Bingham and Reynolds numbers, up to 100 and 5000 respectively, and study systematically the effects of these dimensionless numbers on the flow. The rest of the paper is organised as follows: The associated governing equations are presented in Section 2. The numerical method for solving these equations is briefly described in Section 3, while the results of the numerical experiments are given and discussed in Section 4. Finally, Section 5 summarises the conclusions of this research.

2. Governing equations

The flow is assumed to be steady-state, two-dimensional, incompressible, and isothermal. By scaling the fluid velocity by the lid velocity U , and the pressure and stress by $\mu U/L$, the continuity and momentum equations can be written in the following dimensionless forms:

$$\nabla \cdot \mathbf{u} = 0 \quad (5)$$

$$Re \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \nabla \cdot \boldsymbol{\tau} \quad (6)$$

where \mathbf{u} denotes here the dimensionless velocity of the fluid, p is the dimensionless pressure, and $\boldsymbol{\tau}$ is the dimensionless stress tensor (for the sake of simplicity, we kept the same symbols for the dedimensionalised variables). The dimensionless form of the Papanastasiou constitutive equation becomes

$$\boldsymbol{\tau} = \left[\frac{Bn}{\dot{\gamma}} \{1 - \exp(-M\dot{\gamma})\} + 1 \right] \dot{\gamma} \quad (7)$$

where M is the dimensionless stress growth parameter, defined as:

$$M \equiv \frac{mU}{L} \quad (8)$$

and $\dot{\gamma}$ is now the dimensionless rate of strain tensor. The Papanastasiou regularisation (7) corresponds to a dimensionless apparent viscosity of

$$\eta = \frac{Bn}{\dot{\gamma}} \{1 - \exp(-M\dot{\gamma})\} + 1 \quad (9)$$

The higher the value of M , the better Eq. (7) approximates the actual Bingham constitutive equation, $\boldsymbol{\tau} = [Bn/\dot{\gamma} + 1]\dot{\gamma}$, in the yielded regions of the flow field ($\tau > Bn$), and the higher the apparent viscosity

is in the unyielded regions, making them behave approximately as solid bodies. For practical reasons though, M must not be so high as to cause convergence problems to the numerical methods used to solve the above equations.

Equations (5) – (7) together with the no-slip wall boundary conditions fully determine the flow problem which is solved numerically. In the present study, the direction of motion of the lid is towards the right.

3. Numerical method

3.1. Solution without local grid refinement

Most of the results presented in the following section were computed on Cartesian grids without local grid refinement, consisting of 512×512 square control volumes. Coarser grids are constructed by removing every second grid line from the immediately finer grid. The finite volume method used to solve the governing equations on these grids is described in detail in Syrakos et al. [26]. According to the finite volume methodology, the continuity and momentum equations are integrated over each control volume and the integrals are approximated by algebraic expressions involving the values of the flow variables at discrete points. In the present work, all variables (velocity components, pressure, and viscosity) are stored at control volume centres. Both the convective and viscous fluxes are discretised using 2nd-order accurate central differences. The mass fluxes are discretised using momentum interpolation as described in [50], to suppress spurious pressure oscillations between control volume centres.

The resulting algebraic system is solved using the SIMPLE algorithm, with the only modification being that at the start of every SIMPLE iteration the viscosity is updated according to Eq. (9), using the current estimate of the velocity field. To accelerate convergence, SIMPLE is used in a geometric multigrid framework. Due to the high degree of nonlinearity of the problem, the standard multigrid algorithm fails to converge except at small Bingham numbers, $Bn < 0.5$, as the results in [26] show. To overcome this problem, the modification suggested by Ferziger and Peric [1] has been applied; on coarse grids the viscosity is not updated according to Eq. (9), but it is interpolated (restricted) from the immediately finer grid and held constant within the multigrid cycle. Therefore the viscosity is updated only on the finest grid, which means that the procedure is not purely multigrid, but it has single-grid features. This technique was observed to slow down the multigrid convergence, but it makes the algorithm more robust and capable of achieving convergence up to high Bingham numbers (depending also on the value of M). Other measures that were found necessary in order to achieve convergence are the following: a large number of pre- and post-smoothing steps should be used (4 or more, depending on the value of Bn); a number of additional SIMPLE iterations (e.g. 5 – 20) may have to be performed on the finest grid between multigrid cycles; very small values of the underrelaxation factor for pressure, denoted here by a_p , should be used in the SIMPLE smoother (e.g. 0.01), and in the case of high Reynolds numbers ($Re \geq 2000$) also relatively small values of the underrelaxation factor for velocity, a_u , (e.g. 0.3 – 0.4) should be used; and the coarse grid corrections may have to be underrelaxed by a constant $\alpha_{MG} < 1$ prior to prolongation to the fine grid (usually $\alpha_{MG} \approx 0.9$ suffices). We use W cycles, denoted by $W(\nu_1, \nu_2) - \nu_3$, where ν_1 SIMPLE iterations are performed prior to restriction, ν_2 SIMPLE iterations are performed after prolongation, and ν_3 extra SIMPLE iterations are applied only on the finest grid at the end of each cycle. More details can be found in [26]. For more information on multigrid in general, the reader is referred to [51] or [52].

As noted in [26], the SIMPLE/multigrid procedure becomes less efficient as either Bn or M increase, although the multigrid efficiency is always much higher than that of SIMPLE as a single grid solver. It has been observed that it is useful, or sometimes necessary, to use a good initial guess. This could be, for example, a solution on a coarser grid, or a solution obtained with a smaller value of M , both

of which are more easily computable. In the present work we used mostly the former choice, but the latter choice was also useful in some “difficult” cases, and also led to the following idea: Instead of using a fully-converged lower M solution as the initial guess, start with a very low value of M , say $M = 1$, and progressively increase the value of M every n_M , say, multigrid cycles, n_M being a small constant of the order 1–4, until M obtains its maximum value, beyond which point its value is held fixed until the multigrid cycles converge. This technique can increase the efficiency in some cases, as will be shown in the results section.

3.2. Calculation of the truncation error

The truncation error is the natural measure of the discrepancy between the integrals of the differential equations to be solved and their finite volume approximations. It consists of all the terms of the Taylor series expansions that were truncated in order to obtain the discrete finite volume approximations of the differential equations, and which have the form of products of powers of the grid spacing times higher order derivatives of the flow variables. Grid refinement reduces the truncation error, and therefore an efficient grid refinement strategy is to refine the grid locally where the truncation error is large, instead of applying uniform grid refinement throughout the domain. The truncation error is unknown, but can be approximated using various techniques. In the present study the method described in [50, 30] is utilised, which originates from multigrid theory [51].

Suppose P is a finite volume of a grid with characteristic spacing h , ϕ is the unknown function (there could be more than one, for example in our case we have the two velocity components and pressure), and $N_P(\phi) = 0$ is the equation obtained by integrating the differential equation over P and dividing by its volume, while $N_{h,P}$ is the finite volume discrete approximation to N_h . The truncation error, ζ_P , is defined by

$$\zeta_P = N_P(\phi) - N_{h,P}(\phi_h) \quad (10)$$

where ϕ_h is the vector of the values of ϕ at the centres of the finite volumes of grid h . The equation to be solved, $N_P(\phi) = 0$, is equivalent to $N_{h,P}(\phi_h) + \zeta_P = 0$ but since ζ_P is unknown the finite volume procedure solves $N_{h,P}(\tilde{\phi}_h) = 0$ instead, assuming that ζ_P is small enough to be neglected. The set of all such equations for all finite volumes forms an algebraic system which is solved using SIMPLE/multigrid in the present work. The solution ϕ_h^* thus obtained differs from the exact solution by the discretisation error $\epsilon_h = \phi_h - \phi_h^*$.

If the exact solution ϕ were known, then the truncation error could be calculated from (10). The exact solution is not known, but by using a solution on a finer grid, which is more accurate than $\tilde{\phi}_h$, an estimate of the truncation error can be obtained. This reasoning results in the following formula for estimating the truncation error, as is shown, for example, in [30] or [53]:

$$\zeta_h \approx -\frac{1}{2^p - 1} I_{2h}^h N_{2h}(I_h^{2h} \tilde{\phi}_h) \quad (11)$$

where now ζ_h is the vector of truncation errors in all finite volumes of grid h , N_{2h} is the algebraic operator obtained with the same finite volume discretisation on a grid $2h$ which is twice as coarse as grid h , and I_a^b are interpolation operators which transfer a grid function from grid a to grid b . Also, p is the order of the finite volume approximation, that is, it is the smallest power of the grid spacing that appears among the terms that comprise the truncation error. This term reduces more slowly than the rest with grid refinement, and so at some point it becomes the dominant term of the truncation error; therefore $\zeta_h = O(h^p)$. The present finite volume method is second-order accurate, so $p = 2$. In [50] it is demonstrated that, in the case of Newtonian flows, the present finite volume method works well with

the estimate (11), which converges to the exact truncation error with grid refinement, provided that the restriction operator I_h^{2h} is at least third-order accurate if $p = 2$. If the truncation error estimate is used only as a local grid refinement criterion though, then its estimate need not be very accurate.

As the grid is refined, the truncation error converges to zero at a rate which is proportional to h^p , once the leading term has become much larger than the rest. However, the magnitude of the terms of the truncation error depends not only on the powers of h , but also on the higher-order derivatives of ϕ . If the high-order derivatives are large, a very fine grid may be required for the leading term to become dominant and the truncation error to exhibit its asymptotic rate of convergence. In Bingham flows, these derivatives are discontinuous across the yield surfaces, since they are zero inside the unyielded zones and non-zero outside. If a regularised constitutive equation is used, then the derivatives are continuous, but they attain huge values near the yield surfaces, especially as the Bingham number and the parameter M are increased. Therefore it is expected that the truncation error will be large there. In such cases, the truncation error can be reduced more efficiently by refining the grid locally at the high truncation error regions to counterbalance the high values of the derivatives, rather than using uniform grids. In [30] it was shown that local grid refinement is very efficient in high-Reynolds number flows, which exhibit shear layers with large flow derivatives of high order. In the present Bingham flow case, the flow discontinuities are expected to make the gains from local refinement even more significant.

3.3. Solution with local grid refinement

The local grid refinement scheme adopted here is that described in [29] and [30]. After solving the problem on a given grid, those volumes which fulfil some criterion are marked for refinement. Refinement is performed by subdividing a volume (the *parent*) into four smaller volumes (the *children*) by joining the centre of the parent with the midpoints of its four faces. The volumes are organised into *levels*, corresponding to the number of refinements performed to produce that particular volume. Therefore, if a parent volume is of level k then its four children are of level $k + 1$. Figure 1 shows an example of the organisation of a locally refined grid into levels. When a volume is subdivided, its children are created and added to the data structure, but the parent is also retained in the data structure and not destroyed. Volumes that have children are characterised as *local*; they have no effect on the final solution of the problem, but they are used by the multigrid, or more correctly *multilevel*, procedure to accelerate algebraic convergence. Volumes that do not have children are characterised as *global* and comprise the actual grid where the problem is solved.

The *composite grid* consists of all global volumes of all levels, and it is the grid onto which the differential equations are actually discretised (shown on the left in Figure 1). Each volume is regarded as separated from its neighbouring volumes by its faces, and the momentum and mass fluxes through each face are discretised using central differences. Most volumes have four faces, but some volumes that are located at the interfaces between different grid levels may have more - for example volume P of Figure 2 has six neighbours, and is separated from them by 6 corresponding faces. Despite the fact that, in the current study, all volumes have square shape, the central difference approximations of the fluxes

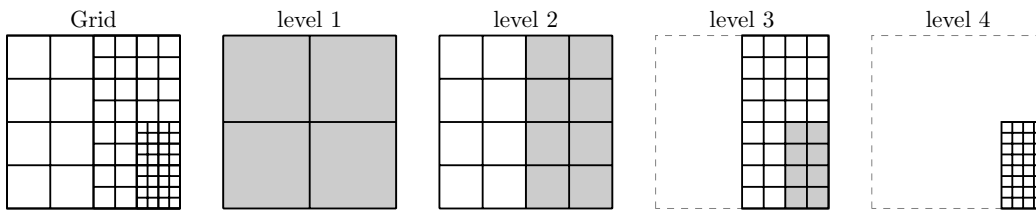


Figure 1: An example of an organisation of a grid into levels. The local part of each level is shown in grey.

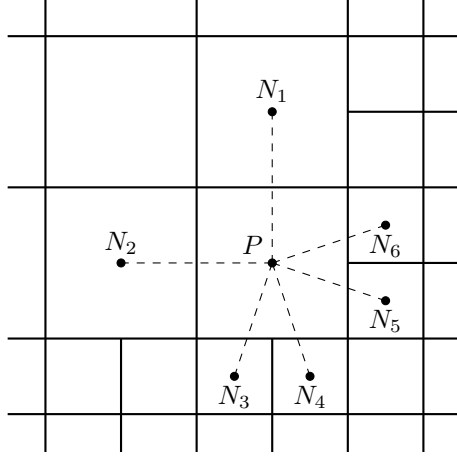


Figure 2: On multilevel grids, volumes which lie at level interfaces may have more than four neighbours.

through the faces which coincide with grid level interfaces would only be first-order accurate. This is because the line segment joining the centres of the volumes on either side of the face is not perpendicular to the face, is not bisected by the face, and does not pass through the face centre. To regain second-order accuracy, additional correction terms are incorporated into the central differencing scheme to account for these geometric irregularities. Full details of the discretisation scheme can be found in [29] or [30].

This discretisation procedure results in a non-linear algebraic system, which is formed using only the global volumes of each level. To solve this system, the local volumes of each level are also used, in order to accelerate algebraic convergence. The equations solved for these volumes are auxiliary equations which approximate the equations of the immediately finer grid, according to the multigrid philosophy. On the contrary, the equations solved for the global volumes of each level are the actual equations of the finite volume discretisation on the composite grid. The algorithm proceeds level-by-level; for example, if V-cycles were used, then the algorithm would proceed from the finest level down to the coarsest one, and then it would move up until the finest level. The fact that some levels do not extend throughout the domain is not a problem, as long as the mass and momentum fluxes through faces that separate global from local volumes are defined appropriately so that when the solution has been attained on the composite grid, and the residuals are zero, the multilevel algorithm does not produce any corrections. The full details of the algorithm can be found in [29].

Using the experience gained in [30], we use the volume integral of the truncation error over each finite volume as the refinement criterion (it is calculated by multiplying the local truncation error estimate by the volume). In particular, after solving the equations on a given grid, this quantity is calculated at each volume of that grid. Then, the volumes are ordered according to the magnitude of this quantity, from highest to lowest. The 20% of the volumes at the top of this list are selected for refinement. This selection procedure is performed for the x - and y -momentum equations, but not for the continuity equation. The union of the two sets of volumes selected through the two momentum equations is the set of volumes which are refined. This results in a new composite grid, where the equations are again solved to obtain a more accurate solution than on the previous grid. The procedure can be repeated to obtain even more refined grids as many times as one wishes.

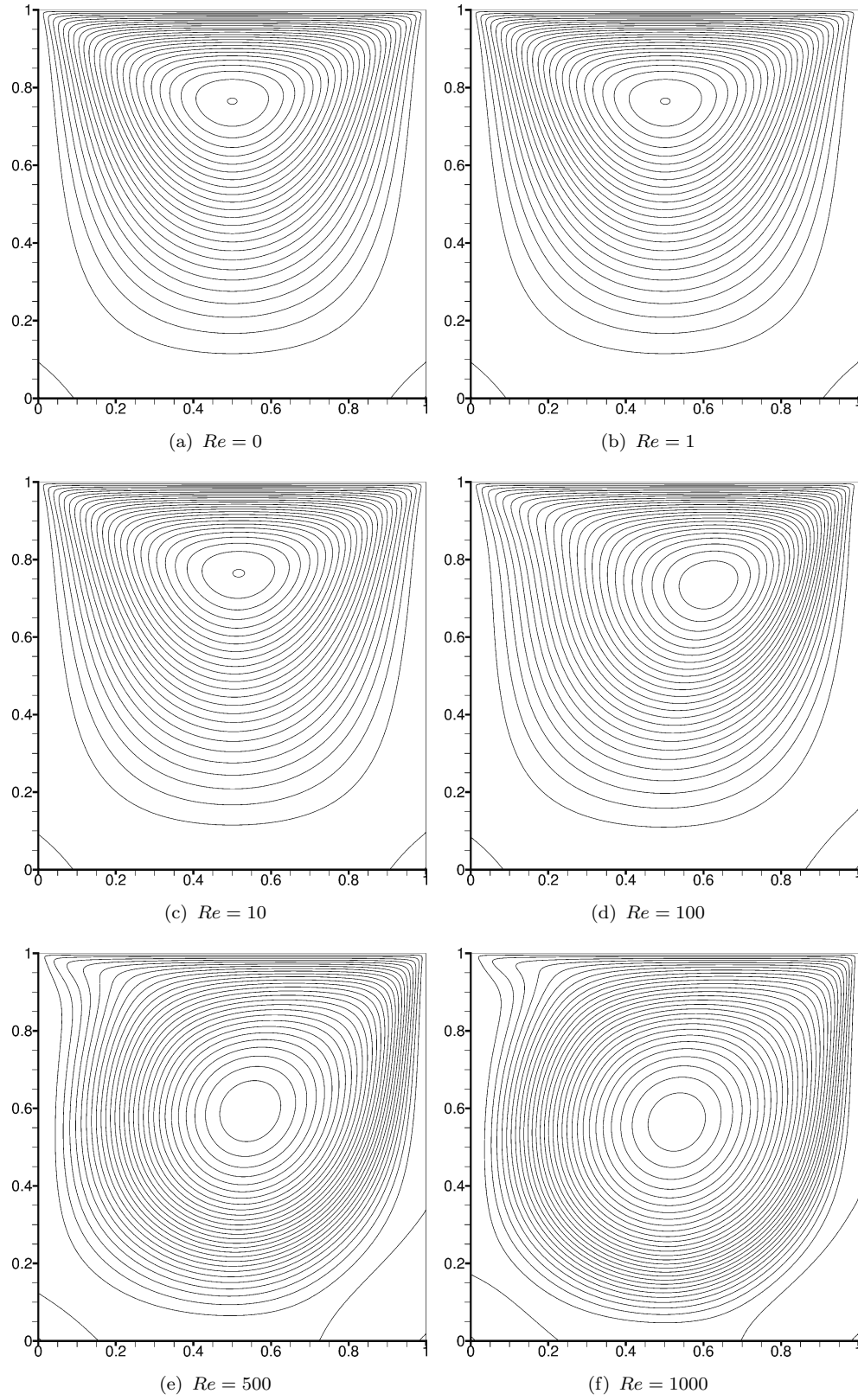


Figure 3: Streamlines in Newtonian flow ($Bn = 0$), plotted at intervals of 0.004 starting from zero.

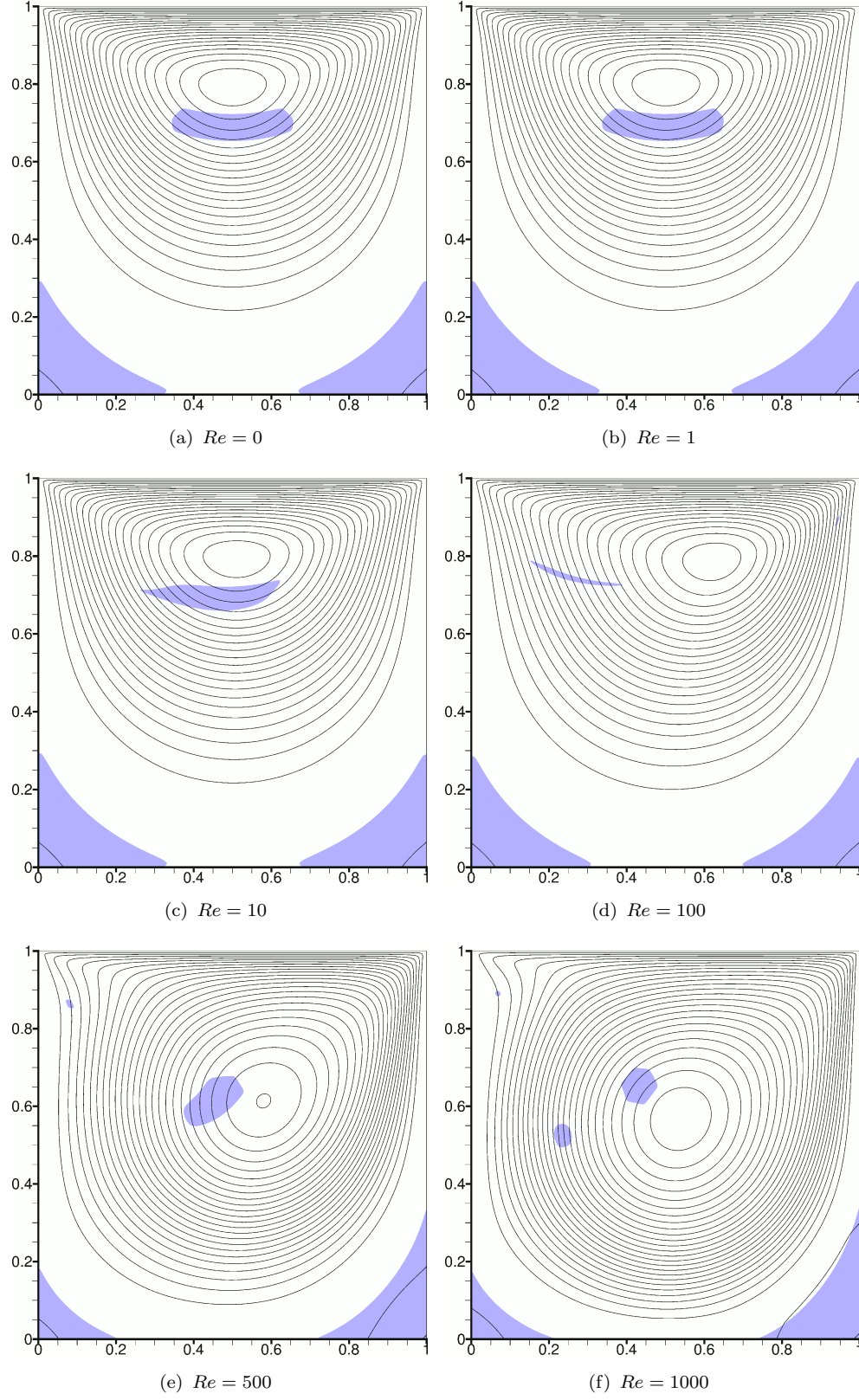


Figure 4: Streamlines in Bingham flow for $Bn = 1$, plotted at intervals of 0.004 starting from zero. Unyielded areas ($\tau < Bn$) are shown shaded.

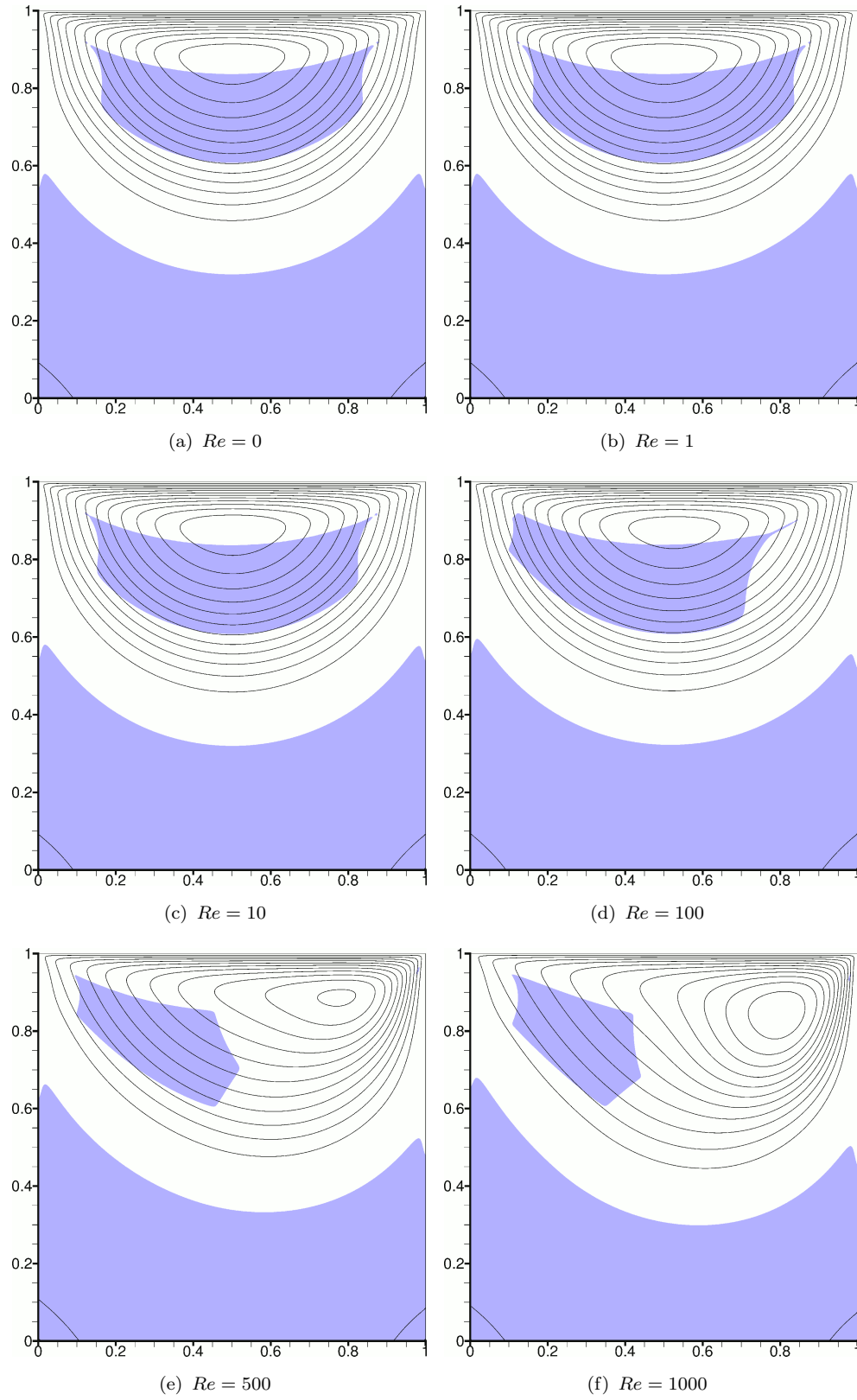


Figure 5: Streamlines in Bingham flow for $Bn = 10$, plotted at intervals of 0.004 starting from zero. Unyielded areas ($\tau < Bn$) are shown shaded.

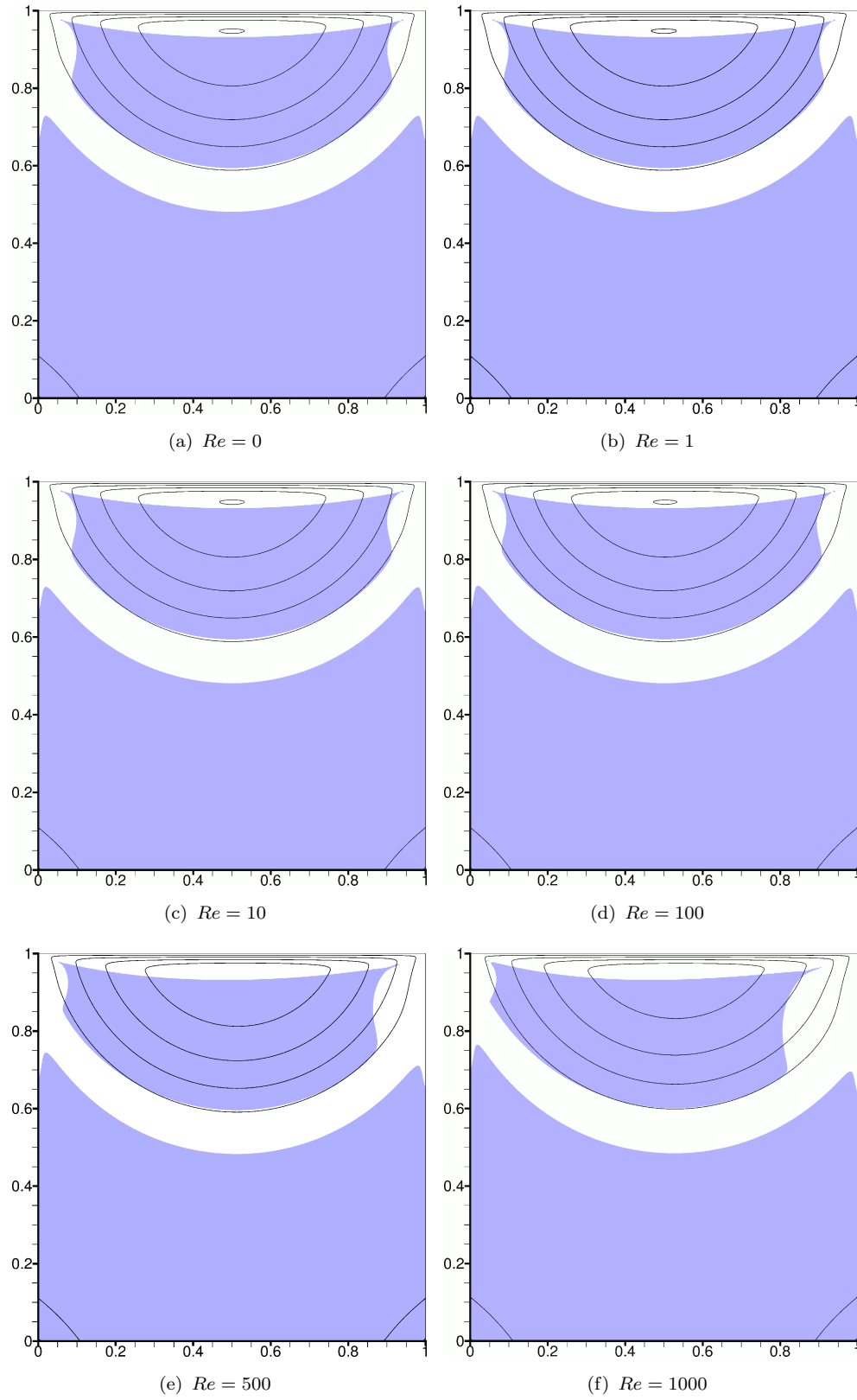


Figure 6: Streamlines in Bingham flow for $Bn = 100$, plotted at intervals of 0.004 starting from zero. Unyielded areas ($\tau < Bn$) are shown shaded.

4. Numerical results

Using the method described in the previous section, the lid-driven cavity problem has been solved for Reynolds numbers up to 5000, and for Bingham numbers up to 100. Unless otherwise stated, the results presented were obtained on the 512×512 uniform grid, using $M = 400$. For $Bn = 100$ and Reynolds numbers other than 1000, a lower value of $M = 200$ was used to shorten the computational time, as the SIMPLE/multigrid method converges very slowly at such a high Bingham number when the value of M is also high. The results of the simulations are presented in subsection 4.1. Then, in subsections 4.2 and 4.3, the accuracy of the results is examined concerning the calculation of the yield surfaces and the velocity field respectively. Finally, in section 4.4 the performance of the SIMPLE/multigrid algebraic solver is discussed.

4.1. Description of the flow field

Figures 3 – 6 describe the flow field as the Reynolds number increases, for $Bn = 0$ (Newtonian flow), 1, 10, and 100, respectively. In the Newtonian case (Fig. 3), the flow field is initially symmetric (Figs. 3(a) - 3(c)) but as the Reynolds number increases the main vortex shifts to the right (Fig. 3(d)), and then towards the centre of the cavity (Figs. 3(e), 3(f)). The same phenomena are observed also in Bingham flow, but they are postponed to larger Reynolds numbers as the Bingham number is increased. This will be discussed in more detail later on.

In the Bingham flow cases, unyielded zones form at the bottom of the cavity because the stresses are low there, due to the distance from the source of motion (the lid). These zones expand as the Bingham number is increased, and leave less space for flow to take place, thus pushing the vortex upwards towards the lid. They are in contact with the side and bottom walls which are motionless, and thus the material in contact is also motionless due to the no-slip boundary condition. This implies, due to the zero rate-of-strain condition within an unyielded zone, that the material is motionless throughout these unyielded zones. Actually, the regularisation method employed allows for weak non-zero deformation rates within unyielded zones, and thus for example one can observe extremely weak vortices at the lower corners, within the unyielded zones. Such features must be regarded as artefacts of regularisation, and discarded in order to get a more accurate picture of the actual Bingham flow.

Figures 4 – 6 show also the existence of one more unyielded zone (in a few cases more than one) which is usually located just below the vortex, or to the left of the vortex when the latter is shifted towards the right. These zones do not touch the cavity walls and are not motionless, as implied from the spacing of the streamlines inside these regions, but move as solid bodies. Since the flow is steady-state, their locations and shapes are fixed, which means that they lose mass on their downstream boundary at a rate equal to that at which they gain mass on their upstream boundary.

As noted, Figs. 3 – 6 reveal that the effect on the flow of increasing the Reynolds number is similar for all Bingham numbers. This can be explored in greater detail with the aid of the plots of the vortex position and strength, Figs. 7 and 8, respectively. Three flow regimes are discernible:

1. Up to a certain Reynolds number the vortex is approximately fixed in space, and its strength is constant ($Re \approx 20$ for $Bn = 0$; $Re \approx 50$ for $Bn = 10$; $Re \approx 500$ for $Bn = 100$).
2. Beyond that Reynolds number, the vortex moves towards the right (in the same direction as the lid), until a second critical Reynolds number is reached ($Re \approx 75$ for $Bn = 0$; $Re \approx 500$ for $Bn = 10$; $Re \approx 5000$ for $Bn = 100$)¹. If the Bingham number is large enough ($Bn \geq 2$) so that

¹For $Bn = 100$, to find this second critical Reynolds number it was necessary to perform a simulation also for $Re = 10000$. We note that these $Re = 10000$ results were obtained on the 256×256 grid due to convergence problems on the 512×512 grid.

the the lower unyielded region has pushed the vortex close to the lid, then this motion of the vortex towards the right is accompanied by a weakening of the vortex, due to geometric restrictions.

3. Beyond the second critical Reynolds number, the vortex moves towards the centre of the cavity. This is accompanied by a strengthening of the vortex.

The main conclusion from these results is that the flow field at a certain combination of Bn and Re numbers resembles to some extent that of any lower Bn number if the Re number is also sufficiently lowered. To investigate this, we calculated the “local Reynolds number”, Re_l , which is defined based on the apparent viscosity $\eta(x, y)$ (9) at each point, instead of the plastic viscosity μ :

$$Re_l(x, y) \equiv \frac{\rho UL}{\eta(x, y)} \quad (12)$$

Figure 9 shows the contours of Re_l for $Re = 1000$ and $Bn = 1, 10$, and 100 (for $Bn = 0$ it is clear that $Re_l = Re = 1000$). Since ρ , U and L are fixed, Re_l is simply proportional to the reciprocal of the effective viscosity. However, the plots allow a comparison between these flows and Newtonian flows of a similar Reynolds number. Indeed, one notices that for $Bn = 1$ (Fig. 9(a)) the local Reynolds number is in the range $200 - 1000$ in most of the cavity, and there is not much difference between this flow field and Newtonian flow at $Re = 1000$ (Fig. 3(f)). For $Bn = 10$ (Fig. 9(b)), Re_l is less than 100 almost everywhere, except near the top and the upper right of the cavity, and the flow resembles a Newtonian flow at $Re = 100$ (Fig. 3(d)), where the vortex has moved towards the right. Finally, at $Bn = 100$ (Fig. 9(c)) it can be seen that Re_l is well below 1 in most of the cavity, and below 10 in most of the yielded

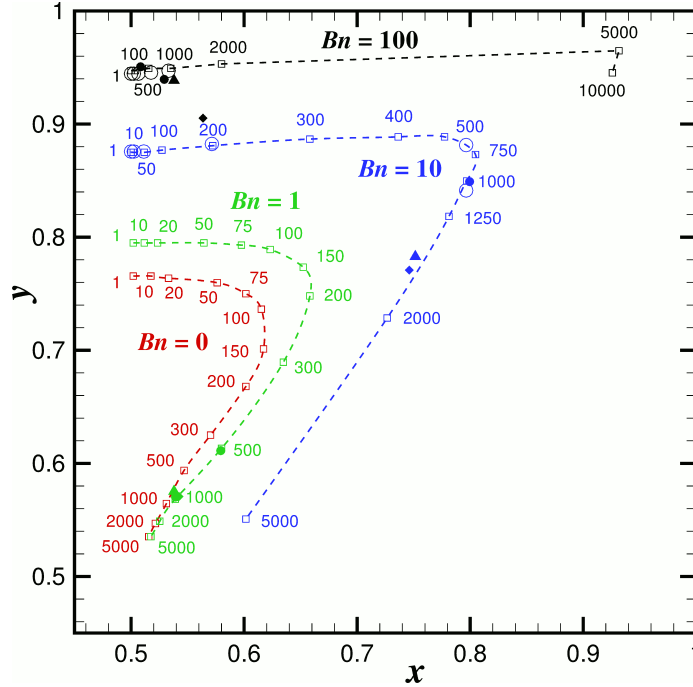


Figure 7: The position of the vortex centre, for various Re and Bn numbers. The results of the present study are shown as empty squares (\square), with the Reynolds number written next to each square. Results of other researchers are also included for comparison: Results of Vola et al. [49] ($Re = 1000$: $Bn = 1, 10, 100$) are indicated by filled triangles (\blacktriangle); results of Elias et al. [45] ($Re = 1000$: $Bn = 1, 10, 100$), are indicated by filled diamonds (\blacklozenge); results of Frey et al. [46] ($Re = 500$: $Bn = 1, 100$; $Re = 1000$: $Bn = 1, 10, 100$) are indicated by filled circles (\bullet); and results of Prashant & Derksen [47] ($Re = 0.5, 10, 50, 200, 600, 1000$: $Bn = 10, 100$), are indicated by empty circles (\circ).

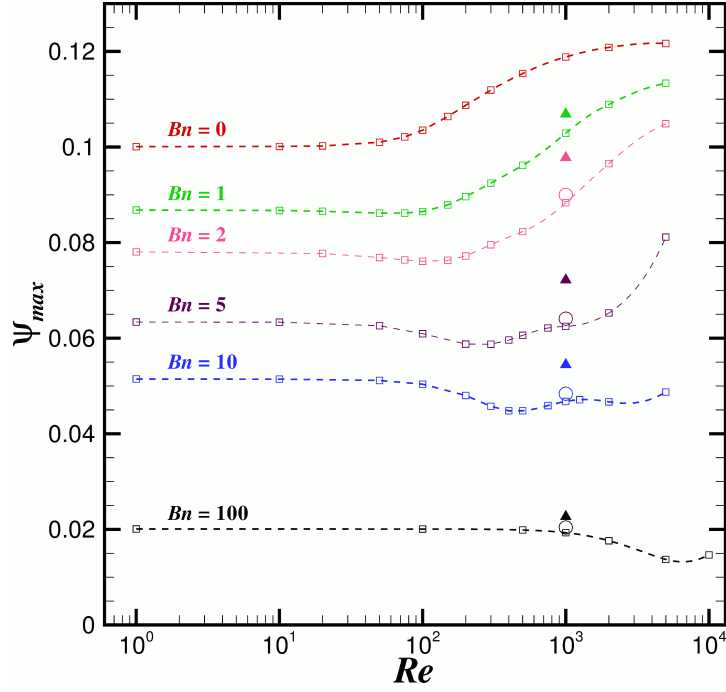


Figure 8: The strength of the vortex, dedimensionalised by $U \cdot L$, as a function of Re for various Bingham numbers. The results of the present study are shown as empty squares (\square). The results of Volá et al. [49] (\blacktriangle) and Prashant & Derksen [47] (\circ) for $Re = 1000$ are also shown.

area - the flow resembles a Newtonian flow at $Re = 1$ or 10 (Figs. 3(b), 3(c)), being nearly symmetric with respect to the vertical centreline.

Finally, we note that Figure 8 shows that increasing the Bingham number causes significant weakening of the flow.

4.2. Accuracy of the yield surfaces

Regularisation methods produce results which do not contain truly unyielded regions. The question is then, how to deduce the unyielded regions from these results. The most common approach is to identify the yield surfaces with the contours of $\tau = \tau_y$ ($\tau = Bn$ in the non-dimensional case). This is the method used in Figures 4 - 6. It is reasonable to assume that the yield surfaces calculated in this manner will converge to the true yield surfaces as $M \rightarrow \infty$, but a theoretical proof of this does not exist, unlike for the velocity field which is known to converge to the exact solution [9]. There do exist a few studies where comparisons against analytical solutions or results obtained with augmented Lagrangian methods have shown that, for the particular test cases studied, yield surfaces calculated with regularisation methods do converge to the true surfaces as the regularisation parameter is increased (e.g. Burgos et al. [54], Dimakopoulos et al. [25]). Unfortunately, the minimum value of the regularisation parameter M which is necessary to accurately predict the yield surfaces is problem-dependent. In cases where the stress is close to the yield stress over a relatively large region, very large values of the regularisation parameter may be needed [9]. In our case, using the SIMPLE/multigrid algebraic solver we were unable to obtain solutions with M larger than about 400, but because the flow domain is confined, the stress variation is rather rapid and therefore good approximations of the yield surfaces can be obtained with low values of M .

Figure 10 shows how the contours $\tau = Bn$ vary as M is increased, for some sample cases. In general, the variation is small, so that one can be confident that the general shape has been captured well. There

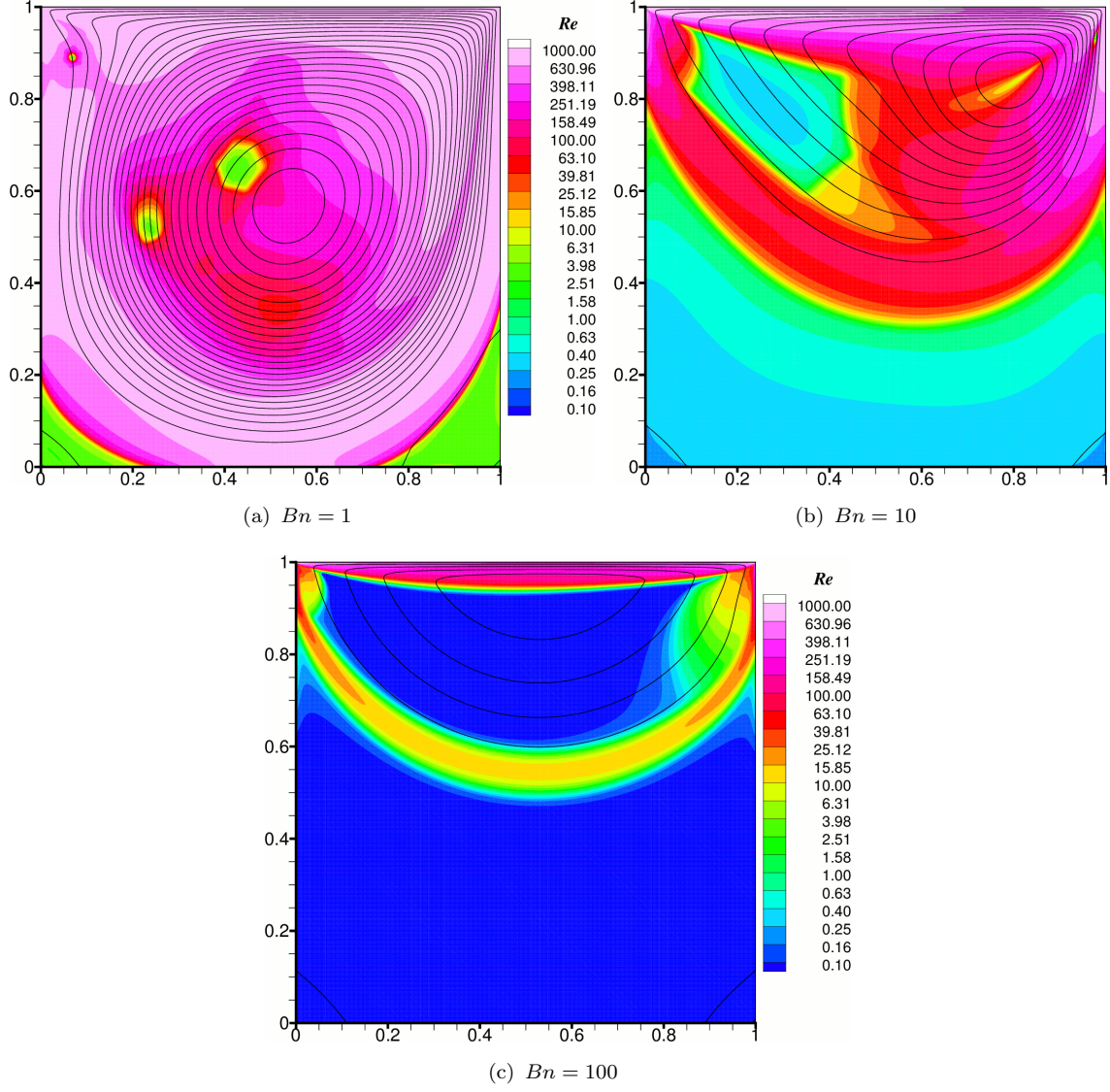


Figure 9: Colour contours of the local Reynolds number Re_l (equation (12)), when $Re = 1000$, for various Bingham numbers. Streamlines are also shown (black lines).

are some inaccuracies in the fine details though, such as the fact that there is a concavity inversion of the $\tau = Bn$ contours where they meet the cavity walls, especially for high Bn , giving the impression that the unyielded zones exhibit “tips” near the walls. Creeping flow results obtained with augmented Lagrangian methods [42, 28, 43, 20], and the limited results for $Re = 1000$ of Vola et al. [49], suggest that this is not a physically correct result. Increasing M improves the results, but the problem has not completely disappeared at the maximum value of $M = 400$ used. For engineering applications such small inaccuracies would probably be unimportant - the errors introduced by the deviation of the chosen mathematical model (e.g. Bingham model) from the behaviour of a real material would be much greater. However, we discuss below a couple of techniques which can be applied at the postprocessing stage to improve the results.

It was suggested by Burgos et al. [54] that the shape of the yield surfaces is described better by contours of $\tau = (1 + \epsilon)Bn$ where $\epsilon \ll 1$ is a small positive number, than by the contours $\tau = Bn$. This is because regularised constitutive equations converge very rapidly to the Bingham equation as τ increases

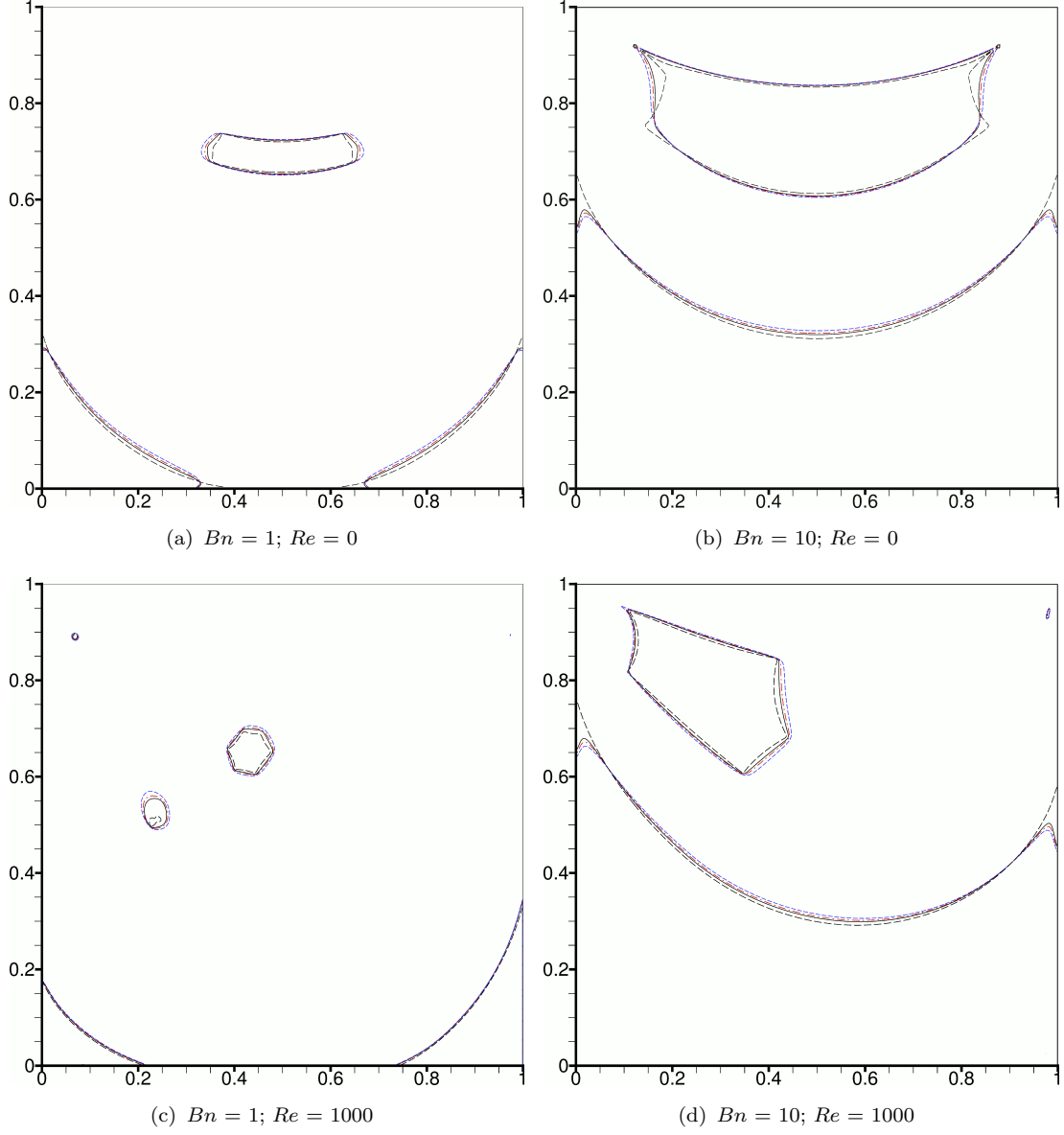


Figure 10: Contours of $\tau = Bn$ calculated with $M = 100$ (blue dashed lines), $M = 200$ (red chained lines), and $M = 400$ (black solid lines). Also shown, with black dashed lines (long dashes) are estimates of the true yield surfaces according to the extrapolation technique of Liu et al. [55].

beyond Bn , but they diverge from the Bingham constitutive equation when τ drops below Bn because the regularised $\tau - \dot{\gamma}$ graph must pass through the origin. Therefore, the contour $\tau = (1 + \epsilon)Bn$ of the Bingham flow field is better approximated by the corresponding regularised contour, than is the contour $\tau = Bn$. And if ϵ is small enough, then the $\tau = (1 + \epsilon)Bn$ contour will not be much different than the $\tau = Bn$ contour. Figure 11 shows that for $Bn=10$ the $\tau = 1.01Bn$ contour in the proximity of the lower unyielded region continues straight up to meet the side walls, in contrast to the $\tau = Bn$ contour which inverts its concavity near the walls, exhibiting a pair of tips. In fact, near the walls it can be seen that the stress is nearly equal to the yield stress ($0.99Bn \leq \tau \leq 1.01Bn$) over relatively large regions, which is precisely the problematic condition described by Frigaard and Nouar [9] under which the yield surface is difficult to compute, and very large M parameters are required. A similar situation appears also at the lower corners of the upper unyielded region of the case $\{Bn = 10, Re = 0\}$.

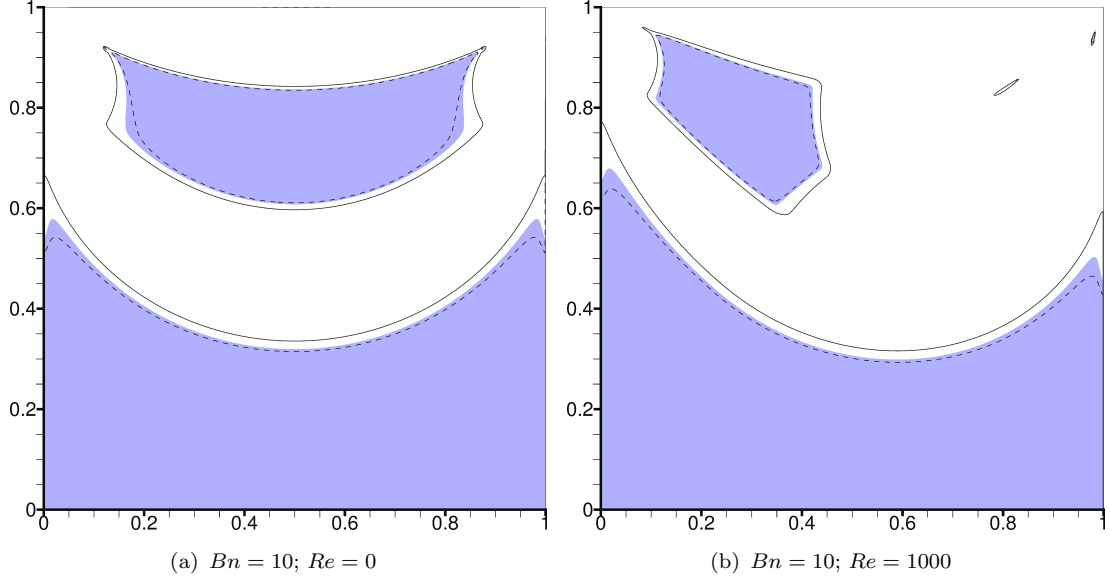


Figure 11: Shown shaded are the unyielded regions ($\tau \leq Bn$), for $Bn = 10$, and $Re = 0$ and 1000. The contours (black lines) correspond to $\tau = 0.99Bn$ (dashed lines) and $\tau = 1.01Bn$ (solid lines).

Liu et al. [55] have proposed an interesting extrapolation procedure to approximate the yield surfaces using a number of solutions corresponding to different values of the regularisation parameter. Their method uses not the stress, but the strain rate. In a real Bingham fluid, when the magnitude of the stress tensor becomes equal to the yield stress then the material is at the onset of yielding, and the strain rate is zero. However, regularisation results in a non-zero critical strain rate $\dot{\gamma}_y$ at the $\tau = Bn$ surface. By writing equation (7) in terms of the tensor magnitudes, and setting $\tau = Bn$, one obtains:

$$\dot{\gamma}_y - Bn \cdot \exp(-M\dot{\gamma}) = 0 \quad (13)$$

This equation can be solved numerically for any given Bn to obtain the corresponding $\dot{\gamma}_y$. Identifying the yield surfaces with the contours of $\dot{\gamma} = \dot{\gamma}_y$ is exactly equivalent to the previous criterion, $\tau = Bn$. Instead, Liu et al. [55] considered the ratio $\tilde{\gamma} \equiv \dot{\gamma}/\dot{\gamma}_y$. At a certain point in the flow field, as M is increased and the exact Bingham flow field is approached, $\dot{\gamma}_y$ tends to zero, while $\dot{\gamma}$ tends either also to zero, if the point is unyielded, or to a specific non-zero value, if the point is yielded. So, the ratio $\tilde{\gamma} = \dot{\gamma}/\dot{\gamma}_y$ tends to infinity in yielded regions, whereas it does not tend to infinity in unyielded regions (where in fact one would expect that $\dot{\gamma}/\dot{\gamma}_y < 1$). Liu et al. [55] noticed in their investigation of the creeping flow of a Bingham material about a sphere that there existed surfaces within the domain where the value of the ratio $\tilde{\gamma}(M) = \dot{\gamma}(M)/\dot{\gamma}_y(M)$ is constant, independent of M . Across such a surface, on one side of the surface the function $\tilde{\gamma}(M)$ increases with increasing M , and thus the material there appears more and more fluid-like, while on the other side of the surface the function $\tilde{\gamma}(M)$ decreases with increasing M (although it may increase again further inside the solid region), and thus the material there appears more and more solid-like. So, Liu et al. [55] suggested that these surfaces coincide with the yield surfaces.

These surfaces can be sought by solving the problem for two values of M and then plotting the difference $\tilde{\gamma}(M_1) - \tilde{\gamma}(M_2)$. The contour $c(M_1, M_2) = \tilde{\gamma}(M_1) - \tilde{\gamma}(M_2) = 0$ is potentially such a surface, because there $\tilde{\gamma}(M_1) = \tilde{\gamma}(M_2)$. To check that this is the sought surface, one can repeat the calculations with one or more different values of M . If for a different value M_3 the contour $c(M_2, M_3) = 0$ coincides with the contour $c(M_1, M_2) = 0$ then this increases the confidence that these contours are the sought

surfaces, because there $\tilde{\gamma}(M_1) = \tilde{\gamma}(M_2) = \tilde{\gamma}(M_3)$. For the present work we used three values of M : 100, 200 and 400. Figure 12 shows the computed surfaces for sample cases. It can be seen that despite the relatively low values of M , the surfaces $c(400, 200) = 0$ and $c(200, 100) = 0$ coincide for the most part, in agreement with the observations of Liu et al. The contours have not converged at the sides of the upper unyielded region of the $\{Re = 0, Bn = 10\}$ case, but in the rest of the domain there is perfect coincidence of the contours, while the tips of the lower unyielded regions have disappeared. In Figure 12, the regions where $c(400, 200) - c(200, 100) < 0$ are shown shaded, as an additional means of investigation, because in yielded regions as $\tilde{\gamma} \rightarrow \infty$, $c(M_1, M_2)$ should be larger than $c(M_2, M_3)$ if $M_1 > M_2 > M_3$ and the parameters are increased in a consistent manner. So, where the shaded regions cross the $c = 0$ contours into the yielded zone, there is uncertainty as to where the yield surface actually lies. This happens for example at the side walls of the upper unyielded region of the $\{Bn = 10, Re = 0\}$ case, where the $c(400, 200) = 0$ and $c(200, 100) = 0$ contours have not yet converged. We note also that there appear $c = 0$ contours inside the upper unyielded regions, near their centres, but they diminish as M increases, and they clearly do not represent yield surfaces. The $c = 0$ contours are plotted also in Figure 10, for direct comparison with the $\tau = Bn$ criterion. To give a better estimate of the yield surface, in the $\{Bn = 10, Re = 0\}$ case the sides of the upper unyielded region have been corrected by joining the lower corners with the points where the contours $c(400, 200) = 0$ and $c(200, 100) = 0$ intersect.

Thus, the technique of Liu et al. [55] appears to yield more accurate yield surfaces. However, for convenience, in the rest of this paper the yield surfaces are taken to be the $\tau = Bn$ (or $\dot{\gamma} = \dot{\gamma}_y$) contours.

Finally, we investigate the effect of grid density on the yield surfaces, calculated as $\tau = Bn$. Figure 13 shows the contours $\tau = Bn$ for the $\{Bn = 10, Re = 1000\}$ case, on various grids. Interestingly, there is observable improvement of the yield surface as the grid density is increased, even up to the 2048×2048 grid: the tips of the boundary of the lower unyielded region move closer to the walls and upwards. The benefit from local refinement is visible, since that particular locally refined grid has approximately the same number of volumes as the 512×512 grid (see the next section for details). Overall though, for the

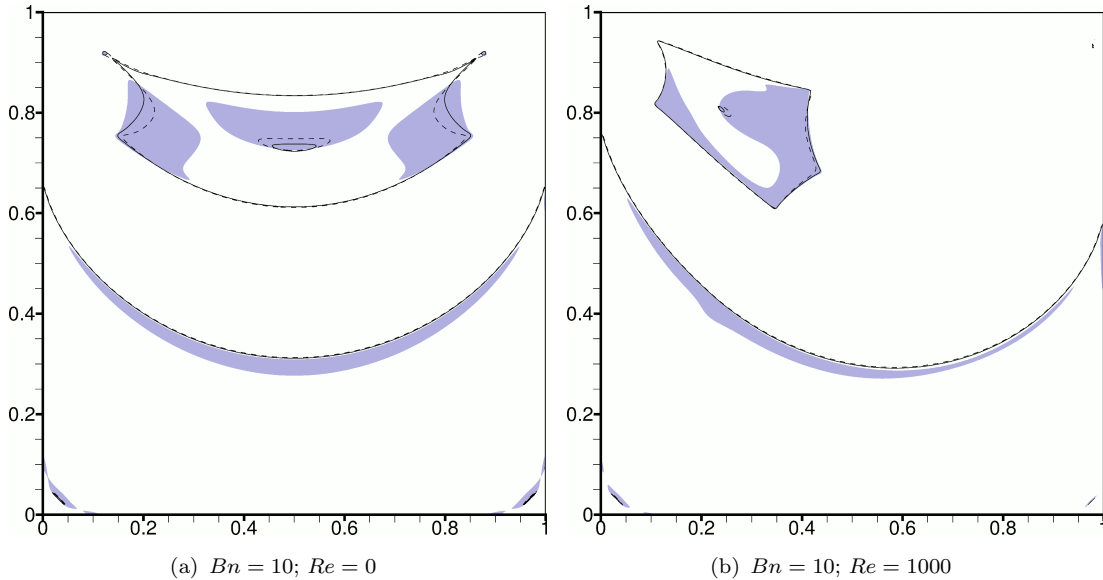


Figure 12: Contours of $c(400, 200) = 0$ (solid lines) and $c(200, 100) = 0$ (dashed lines), which approximate the yield surfaces. See paragraph 4.2 for definition of the function $c(M_1, M_2)$. Shown shaded are the regions where $c(400, 200) - c(200, 100) < 0$, which reveal some uncertainty where they cross the $c = 0$ contours into the yielded region.

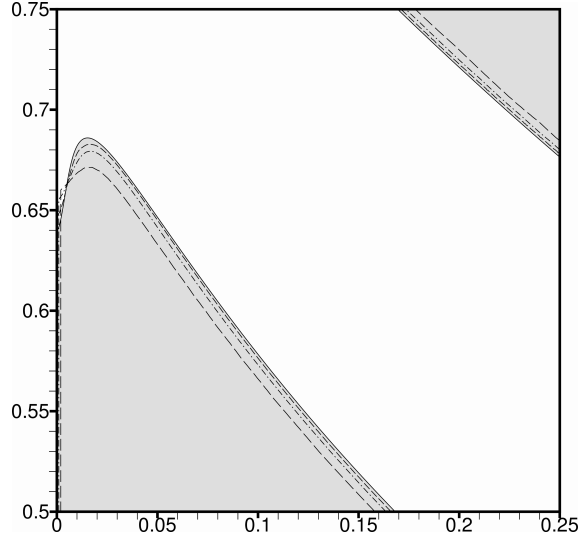


Figure 13: Contours of $\tau = Bn$ on various grids, for the $\{Bn = 10, Re = 1000\}$ case, at a subregion of the domain. The lines correspond to grids 256×256 (long dashes), 512×512 (chained), the locally refined grid of Figure 15(b) (short dashes), and 2048×2048 (solid).

yield surfaces, the effect of M is more significant than the grid density.

4.3. Accuracy of the flow field

Figures 7 and 8 include results by other researchers for validation purposes. In general, there is a good agreement with the present results. There is some discrepancy concerning the vortex position for $\{Re = 1000, Bn = 10\}$ with some of the other publications, while we should also note that Vola et al. [49] predict somewhat stronger vortices. In addition to these results, for creeping flow our previous study [26] showed that the present method produces results that are in very good agreement with the literature.

The rest of this paragraph investigates the effect of Bn , M , and the grid spacing h on the accuracy. To confine the investigation, three cases were selected: $Bn = 1, 10$ and 100 , with Re fixed at $Re = 1000$. To aid the investigation, grid-independent solutions were sought by solving the cases also on finer grids of 1024×1024 and 2048×2048 volumes, and performing Richardson extrapolation (see, for example, [1]), assuming second-order convergence. For the $Bn = 100$ case, results were not obtained on the 2048×2048 grid because this would require a computing time of a few months with our present serial code. Instead, Richardson extrapolation was performed using the 512×512 and 1024×1024 grids. The present finite volume method adopts a cell-centred strategy for storing the variables, which implies that cell centres of different grids do not coincide. Therefore, interpolation is needed in order to compare the solutions of two different grids and perform Richardson extrapolation. We use a third-order accurate interpolation scheme which is described in [30], so that the errors introduced by this interpolation are smaller than those of the finite volume discretisation.

Table 1 summarises most of the results. Use is made of the following norm:

$$|w|_1 = \frac{1}{\Omega} \cdot \sum_{P=1}^N |w_P| \cdot \Omega_P \quad (14)$$

where w is an arbitrary quantity, w_P is the value of this quantity at the centre of cell P of the grid, Ω_P is the volume of cell P , N is the total number of cells of the grid, and Ω is the total volume of the domain. Table 1 contains the following columns:

Table 1: L^1 norms (Eq. (14)) of the flow variables (1st data column), of the discretisation errors ϵ_G^ϕ on different grids G (data columns 2 – 5; LR stands for Locally Refined grid, see Fig. 15), and of the difference $\delta_{M_2}^{M_1}$ of solutions obtained with different regularisation parameters M_1 and M_2 on the 512×512 grid (last two columns). Data columns 6 and 7 (q and q^*) display the order of grid convergence (Eq. (15)), calculated in two different ways (see text). Data in columns 2 – 5 and 8 – 9 are expressed as a percentage of the data in column 1. Unless otherwise stated, $M = 400$.

	$ \phi _1$	$ \epsilon_{128}^\phi _1\%$	$ \epsilon_{256}^\phi _1\%$	$ \epsilon_{512}^\phi _1\%$	$ \epsilon_{LR}^\phi _1\%$	q	q^*	$ \delta_{200}^{100} _1\%$	$ \delta_{400}^{200} _1\%$
$Bn = 1$	# volumes \rightarrow	16,384	65,536	262,144	268,456				
$\phi \equiv u$	1.322×10^{-1}	1.762	0.466	0.119	0.045	1.90	2.00	0.020	0.012
v	1.317×10^{-1}	1.840	0.487	0.125	0.047	1.90	2.00	0.015	0.009
p	5.316×10^{-2}	3.142	0.823	0.213	0.089	1.93	2.04	0.019	0.012
$Bn = 10$	# volumes \rightarrow	16,384	65,536	262,144	284,896				
$\phi \equiv u$	5.555×10^{-2}	3.550	1.166	0.383	0.166	1.60	1.76	0.831	0.437
v	3.260×10^{-2}	4.982	1.639	0.534	0.243	1.60	1.71	1.091	0.577
p	1.052×10^{-2}	7.000	2.443	0.834	0.393	1.50	1.75	1.055	0.603
$Bn = 100$	# volumes \rightarrow	16,384	65,536	262,144	279,280				
$\phi \equiv u$	2.818×10^{-2}	9.618	5.721	2.498	0.673	0.27	1.53	1.856	1.006
v	1.106×10^{-2}	8.948	5.504	2.379	0.826	0.14	1.55	2.927	1.619
p	7.233×10^{-2}	7.831	2.912	1.078	0.448	1.42	1.63	1.294	0.784

- The first data column displays the norm $|\phi|_1$, on the 2048 grid (or the 1024 grid, for $Bn = 100$), where ϕ stands for each of the three main flow variables, u , v and p (one per row).
- The next four columns display the $|\epsilon_G^\phi|_1$ norms of the discretisation errors of the variable ϕ on grid G , computed by comparison against the Richardson extrapolation solution. They are expressed as a percentage of the norm $|\phi|_1$. In the last column, LR stands for the Locally Refined grid shown in Figure 15.
- The next column is the order of grid convergence q defined as (see [1])

$$q \equiv \frac{\log \left(\frac{|\epsilon_{4h}^\phi|_1 - |\epsilon_{2h}^\phi|_1}{|\epsilon_{2h}^\phi|_1 - |\epsilon_h^\phi|_1} \right)}{\log(2)} \quad (15)$$

where the subscripts denote the grid where ϕ was calculated: h , $2h$ and $4h$ are the 512, 256 and 128 grids, respectively. Since the equations were discretised using 2nd-order accurate central differences, q should normally equal 2.

- The column q^* provides an alternative calculation of the order of grid convergence. It uses the solutions on the three finest available grids (512, 1024 and 2018 for $Bn = 1$ and 10; 256, 512 and 1024 for $Bn = 100$) instead of those used by q , and applies a pointwise version of equation (15) (i.e. without the norm) at each cell centre of the coarsest of these grids, to calculate the local order of convergence there. Then, q^* is the average over all grid cells. In this calculation, cells where Eq. (15) returns an undefined or negative result are excluded from the averaging.
- Finally, the last two columns are the norm (14) of the difference $\delta_{M_2}^{M_1} = \phi(M_2) - \phi(M_1)$ of the solutions obtained with different regularisation parameters M_1 and M_2 on the 512 grid, expressed again as a percentage of $|\phi|_1$. As both results are obtained on the same grid, no interpolation is necessary.

The Table has three sections, for $Bn = 1, 10$ and 100 respectively. The header rows of these sections display also the number of volumes of each grid. Of course, the number of volumes of the uniform grids is independent of the Bingham number. The Locally Refined grids slightly vary in volume number (268456, 284896 and 279280 volumes for $Bn = 1, 10$ and 100 , respectively), but in every case the number of volumes is very close to that of the 512×512 grid (262144 vols.), and therefore a direct comparison can be made between the accuracies on the Locally Refined and 512×512 grids.

The results of Table 1 show that for $Bn = 1$ the finite volume method exhibits its nominal 2nd-order convergence, but for $Bn = 10$ the order of convergence drops to 1.60 - 1.75. For $Bn = 100$, q is very small because the differences $(|\epsilon_{4h}^\phi|_1 - |\epsilon_{2h}^\phi|_1)$ and $(|\epsilon_{2h}^\phi|_1 - |\epsilon_h^\phi|_1)$ are nearly equal. However, the fact that ϵ_{512}^ϕ is already quite small relative to ϵ_{256}^ϕ and ϵ_{128}^ϕ means that convergence is accelerating as the grid is refined, and the result $q \approx 0.2$ is too pessimistic. A much more optimistic picture is given by the index $q^* \approx 1.55$ which is calculated on finer grids. It is noticeable that increasing the Bingham number causes also a significant increase of the discretisation error, as a percentage of the solution; it increases by a factor of 3-4 if Bn is raised by an order of magnitude, and this factor appears to increase as the grid is refined.

The fact that increasing the Bingham number causes a deterioration of the convergence rate and an increase in the relative discretisation error can be investigated by examining the truncation error, which is the source of the discretisation error. Figure 14 shows plots of the absolute value of the truncation error of the x -momentum equation, calculated according to estimate (11), for our three selected cases plus the Newtonian case. The truncation error can be seen to increase by several orders of magnitude as the Bingham number increases, which results in the loss of accuracy observed in Table 1. High truncation errors occur mostly in the vicinity of the yield surfaces, where the flow field loses its regularity, but they are also observed elsewhere in the domain. Very high high-order derivatives develop at these locations, giving rise to the high truncation errors, which in turn generate high discretisation errors that are convected and diffused everywhere in the domain.

These results suggest that it would be more efficient to use locally refined instead of uniform grids. Therefore, the selected cases were solved again, starting on the 256×256 uniform grid and allowing two grid refinements, according to the scheme described in Section 3.3. So, for each case the problem had to be solved three times: once on the 256×256 grid, and once after each of the two grid refinements. The final grids obtained are shown on Figure 15. They consist of three levels: the coarsest level has the same density as the 256×256 uniform grid; the intermediate level has the same density as the 512×512 grid; and the finest level has the same density as the 1024×1024 grid. One would then expect on these composite grids an accuracy greater than that of the 256×256 grid, but less than that of the 1024×1024 grid. The value of local refinement is demonstrated in Table 1, where one can see that the error on the locally refined grids is nearly 2.5 times smaller than on the equally-sized (in terms of number of volumes) 512×512 uniform grid. Table 2, which displays pointwise data for the $Bn = 10$ case, shows that this improvement of accuracy occurs everywhere in the domain, including in regions where the composite grid is more coarse than the 512×512 grid (see the points marked in Fig. 15(b)).

The last two columns of Table 1 show the effect of the regularisation parameter M . The change inflicted on the flowfield by changing M from 200 to 400, δ_{400}^{200} , is in every case 0.5-0.6 times that caused by changing M from 100 to 200 (δ_{200}^{100}). The difference δ_{400}^{200} can be viewed as a crude approximation to the error caused by regularisation. For $Bn = 1$, this difference is of the order $\delta_{400}^{200} \approx 0.01\%$ and is much smaller than the discretisation error on the 512×512 grid. For $Bn = 10$ and 100 , δ_{400}^{200} are significantly higher, of the order of 0.5% and 1% respectively, and they are comparable to the discretisation error on the 512×512 grid. These results indicate the necessity of using larger M parameters for simulating flows of higher Bn . In contrast to this result, it has been suggested by researchers who used the Papanastasiou

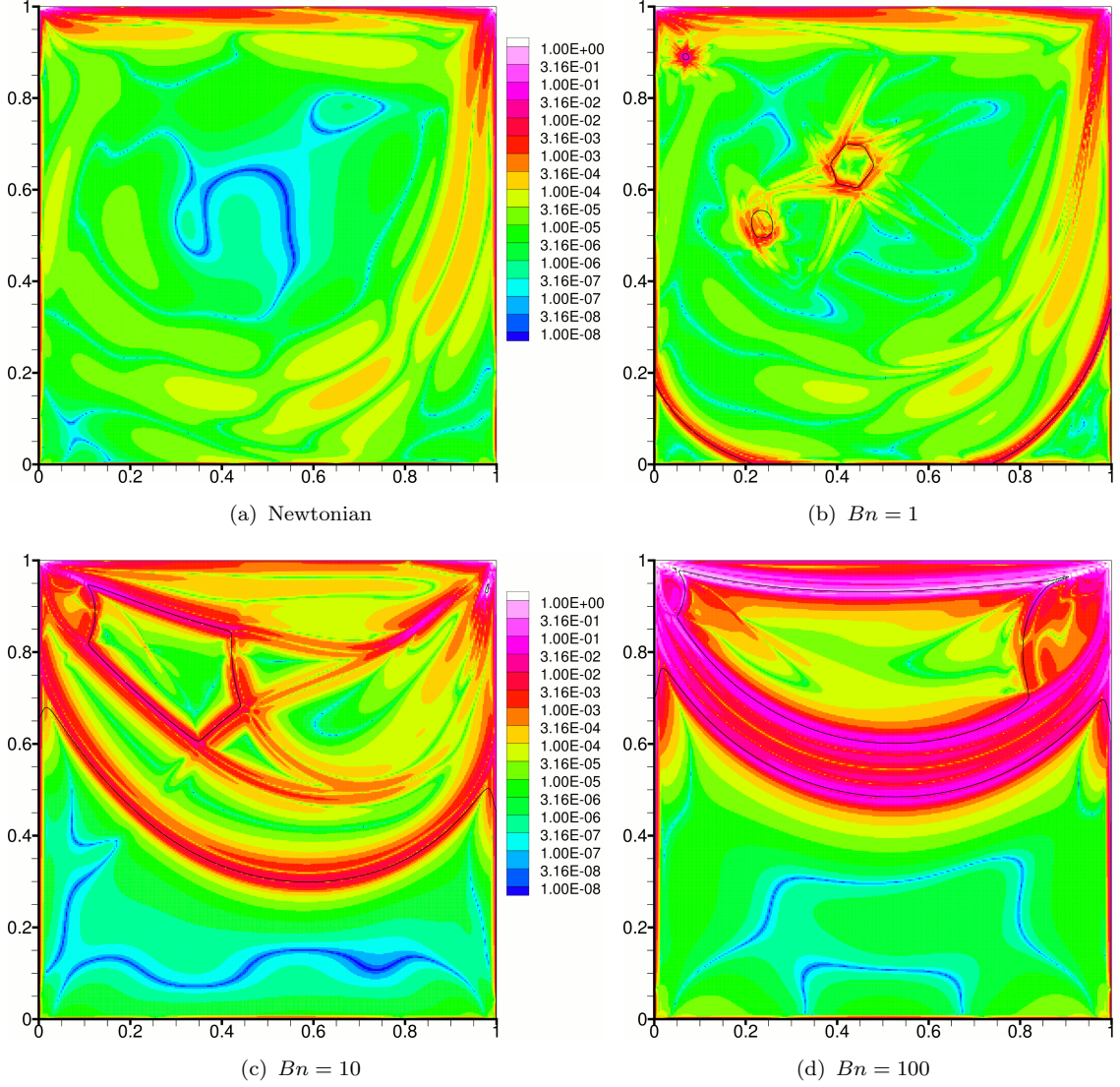


Figure 14: Contours of the absolute value of the truncation error of the x -momentum equation, on the 512×512 grid, according to the estimate (11). The yield lines are shown in black. In all cases, $Re = 1000$.

regularisation, e.g. in [56, 54], that smaller values of M can be used with higher values of Bn , on the basis that the limit of the value of the viscosity η (9) as $\dot{\gamma}$ tends to zero is $M \cdot Bn + 1$. This means that if M is kept constant then in the core of the unyielded regions the viscosity becomes higher as Bn increases, thus providing a better approximation for the unyielded material. However, away from the core, near the yield surface, the approximation of the unyielded material in fact becomes worse as Bn increases, if M is kept constant. This can be seen if one rearranges equation (13) as

$$M = \frac{1}{\dot{\gamma}_y} \ln \left(\frac{Bn}{\dot{\gamma}_y} \right) \quad (16)$$

Therefore, if M is constant, then $\dot{\gamma}_y$ increases with Bn . Figure 16 shows an example where $\dot{\gamma}$ is plotted for two distinct Bn numbers, $Bn = 2$ and $Bn = 50$. It can be seen that for the higher Bn number, $Bn = 50$, $\dot{\gamma}$ is smaller deep into the unyielded zones and larger near the yield lines than for the smaller $Bn = 2$.

Table 2: $\{Re = 1000, Bn = 10\}$ case: Values of u (the x -velocity component) and related discretisation errors ϵ_G^u , obtained on various grids G and expressed as a percentage of u , at selected points of the vertical centreline whose vertical coordinates are shown in the first column. The values of u shown were obtained from the 2048×2048 grid solution with linear interpolation. LR stands for the Locally Refined grid shown in Fig. 15(b), where the selected points are indicated.

y	u	$\epsilon_{256}^u \%$	$\epsilon_{512}^u \%$	$\epsilon_{1024}^u \%$	$\epsilon_{LR}^u \%$
1.000	1.00000	0.00%	0.00%	0.00%	0.00%
0.990	0.78310	0.09%	0.04%	0.01%	0.02%
0.980	0.58701	0.25%	0.07%	0.01%	0.02%
0.960	0.29412	0.60%	0.18%	0.06%	0.04%
0.920	0.04561	5.07%	1.79%	0.47%	0.71%
0.880	-0.02002	8.29%	2.88%	0.79%	1.37%
0.850	-0.03879	0.63%	0.32%	0.12%	0.19%
0.750	-0.05521	0.02%	0.02%	0.01%	0.05%
0.650	-0.06820	0.84%	0.27%	0.07%	0.09%
0.580	-0.07597	0.28%	0.10%	0.03%	0.00%
0.540	-0.07941	0.19%	0.08%	0.03%	0.01%
0.500	-0.07966	0.69%	0.21%	0.05%	0.17%
0.460	-0.06587	1.35%	0.39%	0.09%	0.32%
0.420	-0.04296	3.02%	0.92%	0.22%	0.49%
0.380	-0.02024	7.25%	2.32%	0.57%	0.95%
0.340	-0.00467	16.51%	8.26%	2.42%	4.97%
0.300	-0.00083	4.72%	1.21%	0.27%	0.68%
0.200	-0.00042	0.97%	0.32%	0.07%	0.19%
0.100	-0.00020	0.56%	0.17%	0.03%	0.10%
0.000	0.00000	0.00%	0.00%	0.00%	0.00%

4.4. Algebraic convergence of the SIMPLE/multigrid algorithm

In this subsection some results on the algebraic convergence of the SIMPLE/multigrid algorithm are reported. Figure 17 shows the reduction of the algebraic residuals as a function of the computational effort for $Re = 1000$ and $Bn = 1, 10$, and 100 , with $M = 400$, as typical examples. The ordinate is the the L^∞ -norm of the residual vector of the x -momentum equations,

$$\|r\|_\infty = \max_{P=1,\dots,N} \{|r_P|\} , \quad (17)$$

where r_P is the residual, expressed per unit volume, of the x -momentum equation of volume P and N is the total number of volumes in the grid. The computational effort is measured in equivalent fine-grid SIMPLE iterations. For the multigrid cases, the number of equivalent fine-grid SIMPLE iterations is obtained by multiplying the number of cycles by the number of fine-grid SIMPLE iterations that cost computationally the same as a single cycle. In particular, n_C cycles of type $W(\nu_1, \nu_2) - \nu_3$ cost approximately the same as $n_S = n_C \cdot [2(\nu_1 + \nu_2) + \nu_3]$ SIMPLE iterations on the finest grid (see e.g. [52] on how to calculate the cost of W cycles). For example, one $W(6,6)$ -10 cycle costs the same as 34 fine-grid SIMPLE iterations. It should be noted that the cost of restriction and prolongation is omitted in this calculation, since it is very small compared to the cost of the SIMPLE iterations, especially if one considers that the numbers of pre- and post- smoothing iterations are large, and fine-grid iterations are also carried out between cycles. Therefore, multigrid and single-grid convergence rates are directly comparable in the Figure. The SIMPLE underrelaxation factors were chosen differently in the multigrid and single-grid cases, in order to make the solvers more efficient in each case. For $Bn = 100$, we were unable to make the single-grid algorithm converge on the 512×512 grid, for any choice of underrelaxation parameters.

Figure 17 also shows that the multigrid procedure greatly accelerates the convergence of SIMPLE. One can notice that as the grid becomes finer, the multigrid convergence slows down in general. This

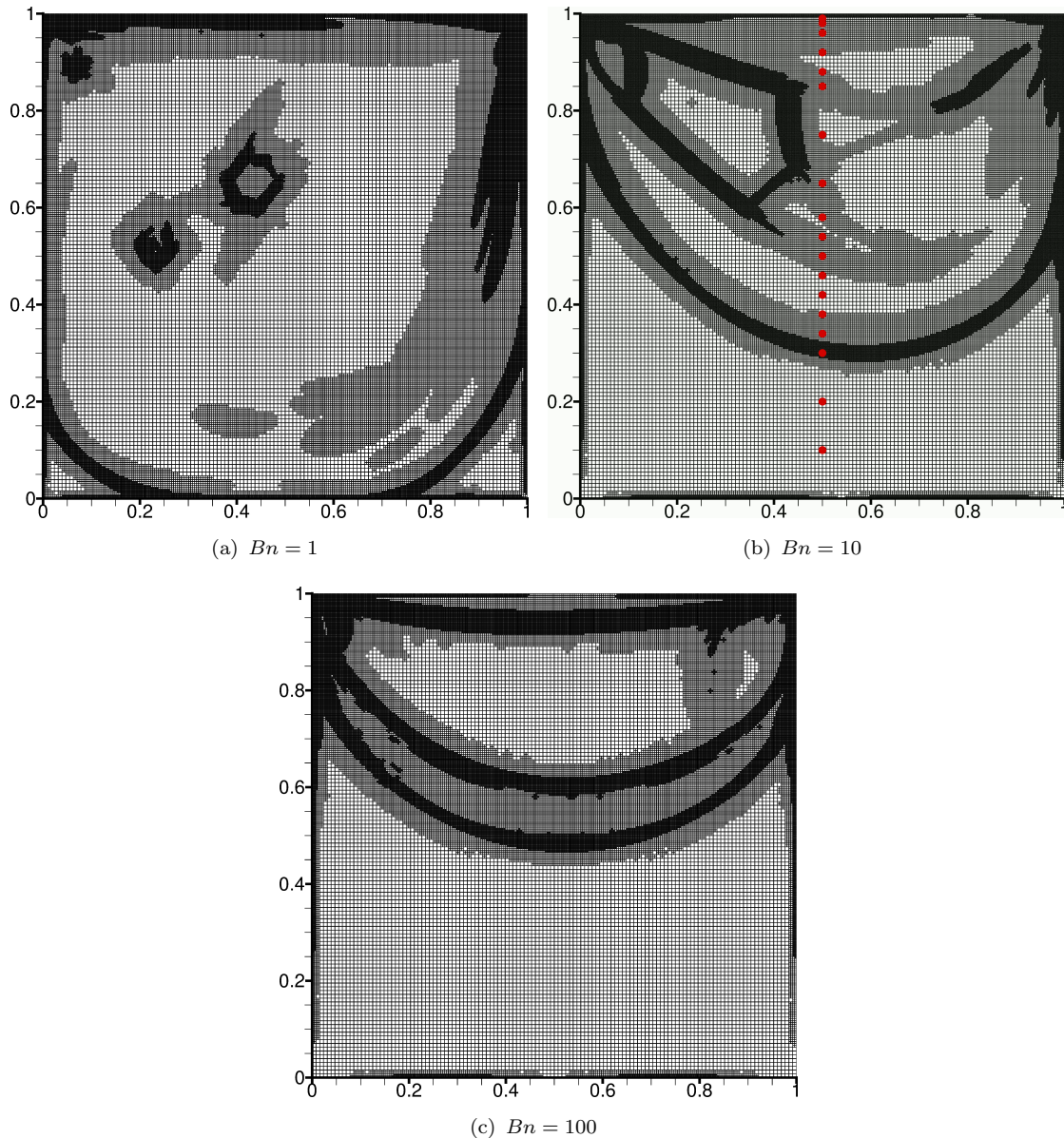


Figure 15: Locally refined grids, for $Re = 1000$ and different Bingham numbers. Actually, for clarity, the grids shown are the *underlying* grids (see [29]), which are twice as coarse as the actual ones used. For $Bn = 10$, the dots along the centreline mark the points that are used in Table 2.

non-typical multigrid behaviour is explained by the fact that the present multigrid method contains single-grid features, as described in Section 3. For $Bn = 10$, on the 256×256 and 512×512 grids the procedure converges fast at the initial stages, due to a good initial guess, but slows down at later stages of iteration. For $Bn = 100$ it is noticeable that convergence is faster on the 512×512 grid than on the 256×256 grid; a possible explanation is that the solution on the 256×256 grid provides a good initial guess for the 512×512 grid, whereas this does not occur on coarser grids. The convergence rates decrease as Bn increases, and are significantly worse than those typically exhibited in Newtonian flows. In fact one may notice that for every ten-fold increase in Bingham number ($Bn = 1$ to $Bn = 10$ to $Bn = 100$) there is roughly also a ten-fold increase in the number of equivalent SIMPLE iterations required.

As mentioned in Section 3, it was observed that it is sometimes advantageous to gradually increase the value of M as multigrid cycles progress, up to the maximum selected value, instead of keeping it at

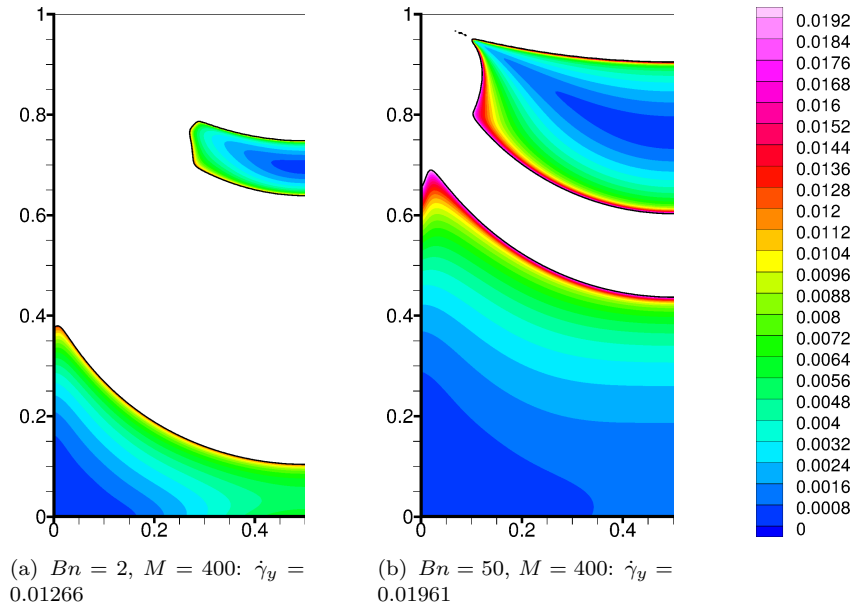


Figure 16: Contours of the magnitude of the rate of strain $\dot{\gamma}$ inside the unyielded zones (yielded regions are shown in white). The Reynolds number is zero, so the flow field is symmetric and only half the domain is drawn. The boundary of the unyielded zones, calculated as $|\dot{\gamma}| = \dot{\gamma}_y$ from equation (13), is marked with a black line.

this value from the start of the calculations. As an example, Figure 18 shows convergence results for $Re = 0$ and $Bn = 20$ with $M = 400$. The “ $M=\text{constant}$ ” curve depicts convergence when $M = 400$ throughout. The other two curves depict the convergence of a procedure where, starting with $M = 1$, after every n_M cycles the exponent M is increased by one. The two curves correspond to $n_M = 2$ and $n_M = 4$. The point where $M = 400$ is reached is marked with vertical dashed lines of the same colour, and from that point onwards the value of M is held fixed at 400. Actually, the residuals shown in Figure 18 prior to the dashed lines ($M < 400$) are not the actual residuals of the exact, $M = 400$, momentum equations, but of the temporary momentum equations with the current value of M (the oscillations are due to the fact that every time M is incremented there is a sharp increase in the residual, since the equations change). But from the dashed lines onwards the residuals can be directly compared among the three curves. It is evident that this technique can bring significant performance gains at no extra cost.

5. Conclusions

A popular method for solving fluid flow problems consists of a combination of a finite volume discretisation and the SIMPLE algebraic solver. Existing codes which use this method are easily extended to solve also viscoplastic flows by using a regularised version of the constitutive equation. Essentially, all that is required is to write a function to calculate the viscosity from the current estimate of the velocity field. The advantages of this approach are that minimum modifications to the code are required, and that all the other features of the code that may have been developed over time are available also for the viscoplastic flows; finite volume / SIMPLE solvers have a long tradition, and many existing codes have a rich set of features including meshing capabilities, numerical schemes, choices of models for different physical phenomena, graphical user interfaces, etc. all of which will be also available for the simulation of viscoplastic flows. On the other hand, regularisation introduces a deviation from the exact equations, which is controlled by a parameter, which therefore plays a role similar to the grid

spacing (and the time step if the flow is transient) in determining the accuracy with which the original problem is approximated. Depending on the application, this deviation may not be very important as the exact constitutive equation is also an approximation to the behaviour of true viscoplastic fluids, and for some materials the regularised constitutive equation may actually be a better approximation of the true behaviour.

In the present work, the capability of the finite volume / SIMPLE method for solving viscoplastic flows was tested by applying it to the very popular lid-driven cavity flow problem, for a range of Bingham and Reynolds numbers, in combination with the regularisation scheme of Papanastasiou [7]. The results showed that both the discretisation errors and the errors due to regularisation increase with the Bingham

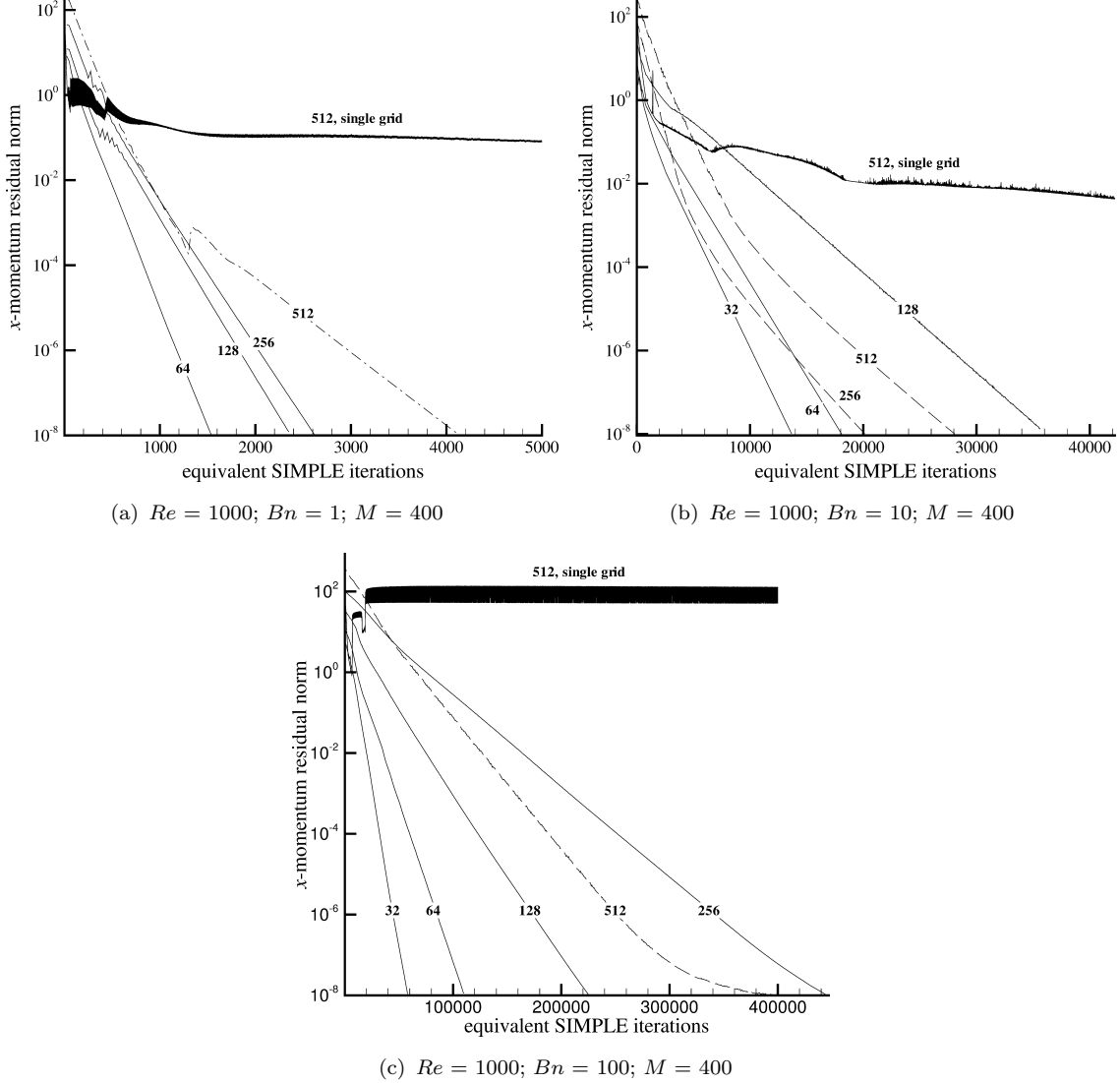


Figure 17: Maximum x -momentum residual per unit volume (17) versus computational effort, for $Re = 1000$, $M = 400$, and $Bn = 1, 10$, and 100 . The results refer to the SIMPLE/multigrid algebraic solver, using the 8×8 as the coarsest grid, except for a single-grid case which is indicated on each figure. The number on each curve indicates the finest grid (n means that the finest grid has $n \times n$ control volumes). The algebraic solver parameters are the following (see Section 3 for definitions): For $Bn = 1$, W(5,5)-5 cycles, $\alpha_{MG} = 1.0$, $a_u = 0.7$, and $a_p = 0.02$ (multigrid) or 0.2 (single grid). For $Bn = 10$, W(6,6)-10 cycles, $\alpha_{MG} = 0.9$, and $\{a_u, a_p\} = \{0.5, 0.02\}$ (multigrid) or $\{0.7, 0.2\}$ (single grid). For $Bn = 100$, W(9,9)-25 cycles, $\alpha_{MG} = 0.9$, and $\{a_u, a_p\} = \{0.4, 0.002\}$ (multigrid) or $\{0.6, 0.1\}$ (single grid). On each grid, the solution of the immediately coarser grid was used as the initial guess.

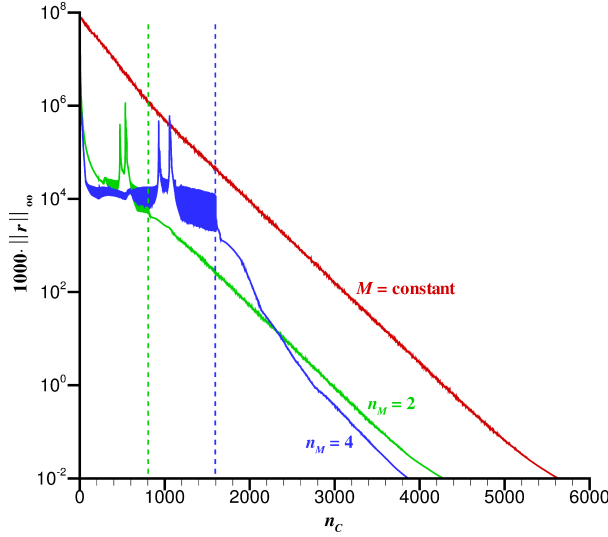


Figure 18: The L^∞ norm of the x -momentum residual plotted against the number n_C of W(6,6)-20 cycles ($a_u = 0.5$, $a_p = 0.01$) on the 256×256 grid, for $Re = 0$, $Bn = 20$, $M = 400$. The solution of 128×128 was used as the initial guess in each case. The red line corresponds to the case that $M = 400$ is used throughout. The other lines depict convergence when M is progressively increased by 1 every n_M cycles, until the maximum of $M = 400$ (marked by dashed lines).

number. The discretisation error increase is due to the truncation error increase in the vicinity of the yield surfaces, because the flow field is nearly discontinuous there and the high-order derivatives of the flow variables attain very large values. Since the high truncation errors are localised, local grid refinement is the most efficient way to reduce them, as the present results verify. On the other hand, the increase of the regularisation error requires that the regularisation parameter is assigned larger values as Bn increases. Unfortunately, with the present method the SIMPLE solver was found unable to cope with regularisation parameters larger than about 400, which is an important weakness of SIMPLE as a viscoplastic flow solver, since this value is rather low according to the literature. Nevertheless, it appears sufficient to produce satisfactory results in the range of Bingham numbers considered here, for the lid-driven cavity problem. This problem is well-behaved in the sense that, due to the confinement of the flow domain, the stress variation is rather rapid and extended regions where the magnitude of the stress is close to the yield stress are not present. Otherwise, very high values of M might be required, as Frigaard and Nouar note [9]. In that case, a stronger solver may be used instead, with all the additional complexity and modifications to the code. Finite Element methods usually use Newton solvers, but the calculation of the Jacobian matrix would be a very difficult task for a finite volume method which uses non-Cartesian grids, as is the case if local grid refinement is applied. However, one could use a Newton-Krylov method thus avoiding explicit calculation of the Jacobian matrix. Such a solver is used for for example by Evans et al. [57] for a phase change problem which also involves fluid and solid regions. In fact they use SIMPLE as a preconditioner, so that the existing SIMPLE routines can be exploited. This is planned to be the subject of a future study.

Another disadvantage of regularisation is that the yield surfaces are not clearly defined. Usually they are identified using the criterion $\tau = \tau_y$, but it is important to place the results under scrutiny, by using different values of M , comparing against $\tau = (1 + \epsilon)\tau_y$ contours, and / or using an extrapolation technique such as that proposed by Liu et al. [55]. These techniques, which are all easily implemented in the post-processing stage without any modifications to the main code, were applied successfully and provided minor corrections to the yield surfaces predicted by the $\tau = \tau_y$ criterion.

Finally we note that regularisation errors can be avoided altogether by using a multipliers method,

with all the additional complexity and programming effort involved. This has been implemented in a Finite Volume context by Vinay et al. [21] and Glowinski and Wachs [20].

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